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SU(3) Chern–Simons field theory in three-manifolds

E. Guadagnini, L. Pilo

Dipartimento di Fisica dell'Università di Pisa, Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Piazza Torricelli, 2, 56100 Pisa, Italy

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Abstract

We present the solution of the non-Abelian SU(3) Chern–Simons field theory defined in a generic three-manifold which is closed, connected and orientable. The surgery rules, which permit us to solve the theory, are derived and several examples of vacuum expectation values of Wilson line operators are computed. The three-manifold invariant associated with the non-Abelian SU(3) Chern–Simons model is defined and its values are computed for various three-manifolds.

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1. Introduction

In this article, we solve the non-Abelian SU(3) Chern–Simons (CS) quantum field theory defined in a generic three-manifold $\mathscr{M}$ which is closed, connected and orientable. Each manifold $\mathscr{M}$ of this type admits a ("honest") surgery presentation given by Dehn surgery on the three-sphere S^3. Each surgery instruction is represented by a framed unoriented link $\mathscr{L}$ in S^3, which is called the surgery link. We shall derive the field theory rules corresponding to a generic surgery instruction. In this way, the expectation values of the gauge invariant observables in $\mathscr{M}$ can be expressed in terms of the expectation values in S^3.

The main features of the SU(3) CS theory in S^3 have been analyzed in Ref. [1]. The present paper is the continuation of the article [1]. Several results obtained in Ref. [1] will be used in our construction of the surgery operator. Our

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work is organized as follows. Firstly, we shall construct the field theory operator which represents surgery when the gauge group is SU(3). Then, several examples of Wilson line expectation values defined in various three-manifolds will be examined. Finally, the three-manifold invariant which represents the value of the improved partition function will be defined; explicit non-trivial three-manifolds will be considered and the value of the associated invariant will be computed.

The construction of the surgery operators for the CS theory with a generic (simple compact Lie) gauge group G is also considered.

2. Dehn surgery

In this section we recall some basic definitions of surgery on three-manifolds. Let us consider surgery operations in S^3. A Dehn surgery performed along a knot $\mathscr{L}$ in S^3 consists of

(1) removing the interior $\mathring{N}$, of a tubular neighbourhood N of the knot $\mathscr{L}$, from S^3;
(2) considering $S^3-\mathring{N}$ and N as distinct spaces whose (distinct) boundaries $\partial(S^3-\mathring{N})$ and ∂N are tori;
(3) gluing back N and $S^3-\mathring{N}$ by identifying the points on their boundaries ∂N and $\partial(S^3-\mathring{N})$ according to a given homeomorphism $h:\partial N\to\partial(S^3-\mathring{N})$.

The knot $\mathscr{L}$ and the "gluing" homeomorphism h completely specify the surgery operation and the resulting manifold is indicated by

$$\mathscr{M}=(S^3-\mathring{N})\bigcup_h N\,. \tag{2.1}$$

Actually, the manifold (2.1) depends [2], up to homeomorphism, only upon the homotopy class of $h(\mu)$ in $\partial(S^3-\mathring{N})$, where μ is a meridian of N. The surgery is characterized then by the knot $\mathscr{L}$ and by a closed curve $Y\in\partial N$ representing $h(\mu)$. The convention, introduced by Rolfsen [2], which is used to codify the surgery instruction is the following. The class $[Y]\in\pi_1(\partial N)$ is written as

$$[Y]=a\cdot[\lambda]+b\cdot[\mu]\,, \tag{2.2}$$

where the generators λ and μ are the longitude and the meridian of a Rolfsen basis [2,3] in ∂N. Since Y is a knot in ∂N, the integer coefficients a and b appearing in eq. (2.2) are relatively prime. The ratio

$$r=b/a \tag{2.3}$$

is called the surgery coefficient. In conclusion, the surgery instruction is specified simply by the knot $\mathscr{L}$ in S^3 and by the rational surgery coefficient r.

The knot $\mathscr{L}$, along which surgery is performed, is not oriented and Y also is not oriented. Therefore, in a fixed Rolfsen basis, the coefficients a and b appearing in eq. (2.2) possess an overall ambiguity in their signs, depending on the choice of the orientation of Y. Similarly, for fixed orientation of Y, a different choice of the Rolfsen basis modifies both signs of the coefficients a and b. This ambiguity does not affect the resulting manifold obtained by surgery and, in fact, this ambiguity disappears in the surgery coefficient r.

When $a=0$, necessarily $b=\pm 1$ and the surgery coefficient is indicated by $r=\infty$. In this case, Y is a meridian of N. This means that the image, under the gluing homeomorphism h, of the meridian μ of N is ambient isotopic with μ itself. Therefore, the resulting manifold is just S^3. In conclusion, the surgery instruction specified by an arbitrary knot $\mathscr{K}$ with surgery coefficient $r=\infty$ corresponds to the identity.

Clearly, the surgery operation of removing and sewing a solid torus can be repeated several times. Therefore, a general surgery instruction consists of an unoriented link $\mathscr{L}$ in S^3, called the surgery link, with given surgery coefficients $\{r_i\}$ assigned to its components $\{\mathscr{L}_i\}$.

For example, when $\mathscr{L}$ is the unknot with surgery coefficient $r=b/a$, the resulting space is homeomorphic with the lens space $L(b, a)$. In particular, the unknot with surgery coefficient $r=0$ corresponds to $S^2 \times S^1$. If $\mathscr{L}$ coincides with the Borromean Rings with all the surgery coefficients equal to $+1$, the resulting manifold is the icosahedral space or Poincaré manifold $\mathscr{P}$. The Borromean Rings with surgery coefficients $r_i=0$, for $i=1, 2, 3$, represent $S^1 \times S^1 \times S^1$.

Different surgery instructions not necessarily correspond to different manifolds. To be more precise, two manifolds associated with different surgery instructions are homeomorphic if and only if the two surgery instructions are related [2,4] by a finite sequence of Rolfsen moves.

A Rolfsen move of the first type is quite obvious: it states that one can add or eliminate a component of the surgery link $\mathscr{L}$ with surgery coefficient $r=\infty$. A Rolfsen move of the second type describes the effects of an appropriate twist homeomorphism $\tau_\pm$ acting on a solid torus which contains part of the surgery link. Let $\mathscr{L}$ be a surgery link in which one of its components, say $\mathscr{L}_1$, is the unknot with surgery coefficient r_1. This means that all the remaining components $\{\mathscr{L}_j\}$ (with $j\neq 1$) of $\mathscr{L}$ belong to the complement solid torus $\mathscr{V}_1$ of $\mathscr{L}_1$ in S^3. Under a twist homeomorphism $\tau_\pm$ of $\mathscr{V}_1$, the component $\mathscr{L}_1$ is not modified, i.e. $\mathscr{L}'_1=\mathscr{L}_1$. The remaining components $\{\mathscr{L}_j\}$ are transformed under the twist $\tau_\pm$, $\tau_\pm : \mathscr{L}_j \to \mathscr{L}'_j$, according to the rules illustrated in Refs. [2,3]. Furthermore, the surgery coefficients also are modified. One can show [2] that the new surgery coefficients are

$$r'_1 = \frac{1}{(1/r_1)\pm 1}, \tag{2.4}$$

$$r'_j = r_j \pm [\chi(\mathscr{L}_j, \mathscr{L}_1)]^2 \quad \text{for } j\neq 1, \tag{2.5}$$

where $\chi(\mathscr{L}_j, \mathscr{L}_1)$ is the linking number of $\mathscr{L}_j$ and $\mathscr{L}_1$. Thus, we have two surgery instructions which are described, respectively, by

(i) the surgery link $\mathscr{L}$ with surgery coefficients $\{r_i\}$;
(ii) the surgery link $\mathscr{L}'$ with surgery coefficients $\{r'_i\}$.

The surgery instructions (i) and (ii) are said to be related by a Rolfsen move of the second kind and describe homeomorphic manifolds.

The three surgery instructions shown in Fig. 2.1 describe the manifold $S^2 \times S^1$. The first and the second instructions are related by a Rolfsen move of the second kind, whereas the second and the third instructions are related by a Rolfsen move of the first kind. The surgery instructions shown in Fig. 2.2 are related by Rolfsen moves and correspond to the Poincaré manifold $\mathscr{P}$.

Two different surgery instructions, which are related by a finite sequence of Rolfsen moves, are called equivalent. The set of all possible surgery instructions can be decomposed into classes of equivalent instructions and a function defined on these equivalence classes is called a three-manifold invariant.

It should be noted that the finite sequences of Rolfsen moves actually correspond to the set of orientation-preserving self-homeomorphisms of a given three-manifold. If we wish to include self-homeomorphisms which are not orientation-preserving, we simply need to add the inversion of S^3 (or mirror-reflection) as an admissible move.

Let $\mathscr{M}$ be the manifold described by the surgery link $\mathscr{L}$. The ambient isotopy class of a given framed oriented link in $\mathscr{M}$ will be represented by a framed oriented link L in the complement of $\mathscr{L}$ in S^3.

The Lickorish (or Fundamental) Theorem [5] states that every closed, orientable and connected three-manifold can be obtained by surgery in S^3; moreover one may always find such a surgery presentation in which the surgery coefficients are all ± 1 and the individual components of the surgery link are unknotted. Let us denote by S_+ and S_- the elementary surgery operations in S^3 corresponding to the instructions described by the unknot U in S^3 with surgery coefficient $r = +1$ and $r = -1$ respectively. The three-manifold obtained according to a single surgery instruction S_+ or S_- is homeomorphic with S^3. By combining several sur-

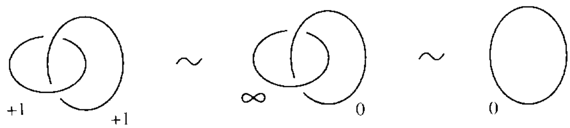


Fig. 2.1. Surgery instructions related by Rolfsen moves.

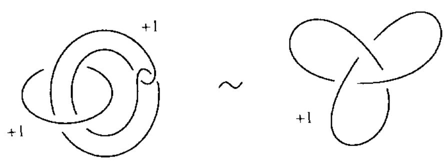


Fig. 2.2. Equivalent surgery instructions.

geries $S_\pm$, any three-manifold $\mathscr{M}$ can be obtained.

Let us introduce the so-called "honest" surgeries [2,4,5]. It is not difficult to prove that one can always find a "honest" surgery presentation of a given three-manifold: this simply means that all the surgery coefficients $\{r_i\}$ are integers. When the ratio r, appearing in eq. (2.3), is an integer, one can take $a=1$ and $b=r$ in eq. (2.2). In this case, the curve Y is a longitude of N and can be interpreted as a framing of the surgery knot $\mathscr{Z}$. The linking number $\chi(\mathscr{Z}, Y)$ of $\mathscr{Z}$ and Y is given precisely by the coefficient b, see eq. (2.2), and then

$$\chi(\mathscr{Z}, Y)=r . \tag{2.6}$$

Therefore, a surgery link $\mathscr{L}$ with integer surgery coefficients $\{r_i\}$ can be represented by a framed link $\mathscr{L}$ in which the linking number of the component $\mathscr{L}_i$ and its framing $\mathscr{L}_{if}$ is equal to r_i.

Let us now consider the set of all possible instructions corresponding to "honest" surgeries. Two such different instructions describe homeomorphic manifolds if and only if they are related [4] by a finite sequence of Kirby moves. A Kirby move is the analogue of a Rolfsen move and can be defined as follows. Suppose that one component, say $\mathscr{L}_1$, of $\mathscr{L}$ is the unknot with framing $\mathscr{L}_{1f}$ such that $\chi(\mathscr{L}_1, \mathscr{L}_{1f})=\pm 1$. Then, this component can be eliminated provided that we perform a twist $\tau_\mp$ on the complement solid torus $\mathscr{V}_1$ of $\mathscr{L}_1$ in S^3. In general, let us consider a Kirby move when a real link L is also present in the manifold $\mathscr{M}$. In this case, the twist $\tau_\mp$ of solid torus $\mathscr{V}_1$ clearly acts on all the remaining components of $\mathscr{L}$ and, simultaneously, on the link L.

By using eqs. (2.4) and (2.5), entering the definition of the Rolfsen moves, and the transformation property [2] of framings under twist homeomorphisms, it is easy to prove that the invariance under Kirby moves is in fact equivalent to the invariance under Rolfsen moves.

3. Elementary surgeries

In order to solve the CS theory in a generic three-manifold $\mathscr{M}$, we need to compute the expectation values of Wilson line operators in $\mathscr{M}$. As we have already mentioned, each link in $\mathscr{M}$ can be represented by a link in the complement of $\mathscr{L}$ in S^3, where $\mathscr{L}$ is a surgery link corresponding to $\mathscr{M}$. According to the Lickorish Theorem, $\mathscr{L}$ can be taken to be a collection of unknots with surgery coefficients ± 1. Thus, we only need to consider the meaning of the single elementary surgery operations S_+ and S_-.

The surgery instruction associated with $S_\pm$ is represented by the unknot U in S^3 with surgery coefficient $r=\pm 1$. From the invariance under Kirby moves, it follows that S_+ (S_-) can be interpreted [3] as the generator of a left-handed twist τ_- (right-handed twist τ_+) of the complement solid torus of U in S^3. Therefore, for any given framed link L in the complement of U in S^3, the action of $S_\pm$ is given by

$$S_\pm : L \to L^{(\mp)} , \tag{3.1}$$

where $L^{(\mp)}$ is the image of L under a twist homeomorphism $\tau_\mp$ of the complement solid torus of U in S^3. So, under surgery $S_\pm$, the observables of the CS theory transform as

$$S_\pm : \langle W(L) \rangle|_{S^3} \to \langle W(L^{(\mp)}) \rangle|_{S^3} . \tag{3.2}$$

At this point, our strategy is to find [3] field theory operators $\tilde{W}(U; \pm 1)$ which represent the surgery operations $S_\pm$ according to

$$\langle W(L^{(\mp)}) \rangle|_{S^3} = \langle W(L) \tilde{W}(U; \pm 1) \rangle|_{S^3} / \langle \tilde{W}(U; \pm 1) \rangle|_{S^3} . \tag{3.3}$$

For the quantum CS field theory in S^3, with fixed integer k, the physically inequivalent gauge invariant quantum numbers associated with a knot (or a solid torus) are given [1] by the elements of the reduced tensor algebra $\mathscr{T}_{(k)}$. Let us denote by $\{\psi_i\}$ the elements of the standard basis of $\mathscr{T}_{(k)}$, where the collective index i runs from 1 to the dimension of the reduced tensor algebra and ψ_1 denotes the identity element in $\mathscr{T}_{(k)}$.

Let us recall that any gauge invariant observable $\mathscr{O}(C)$ associated with the knot C admits [1] a decomposition

$$\mathscr{O}(C) = \sum_\rho \xi'(\rho) W(C; \chi[\rho]) , \tag{3.4}$$

where $\{\xi'(\rho)\}$ are numerical (complex) coefficients. In eq. (3.4), the sum is over the inequivalence irreducible representations of the gauge group. The Wilson line operators, entering eq. (3.4), are defined for the knot C with a given choice of its framing and its orientation. For fixed integer k, different elements $\chi[\rho]$ and $\chi[\rho']$ of the tensor algebra $\mathscr{T}$ not necessarily represent physically inequivalent colour states [1]. In terms of the physically inequivalent colour states, eq. (3.4) becomes

$$\mathscr{O}(C) = \sum_i \xi(i) W(C; \psi_i) , \tag{3.5}$$

where $\{\psi_i\}$ are the elements of the standard basis of the reduced tensor algebra $\mathscr{T}_{(k)}$ and the coefficients $\{\xi(i)\}$ are linear combinations of $\{\xi'(\rho)\}$. Therefore, the elementary surgery operators $\tilde{W}(U; \pm 1)$ can be written as

$$\tilde{W}(U; \pm 1) = \sum_i \phi_\pm(i) W(U; \psi_i) , \tag{3.6}$$

where, in our convention, the unknot U has preferred framing and a fixed orientation. Now, our purpose is to determine the coefficients $\{\phi_\pm(i)\}$. Let us consider a generic framed L which does not intersect U. This means that L belongs to the complement solid torus $\mathscr{V}$ of U in S^3. According to eq. (3.5), $W(L)$ can always be written as

$$W(L)=\sum_j \xi(j)\,W(C;\psi_j)\ , \tag{3.7}$$

where $W(C;\psi_j)$ is the Wilson operator associated with the oriented core C of $\mathscr{V}$ with preferred framing. Let $L^{(\pm)}$ ($C^{(\pm)}$) be the image of L (C) under a twist homeomorphism $\tau_\pm$ of $\mathscr{V}$. From eq. (3.7) it follows that

$$W(L^{(\pm)})=\sum_j \xi(j)\,W(C^{(\pm)};\psi_j)\ . \tag{3.8}$$

The vacuum expectation value of both sides of this equation gives

$$\langle W(L^{(\pm)})\rangle|_{S^3}=\sum_j \xi(j)\,q^{\pm Q(j)}E_0[j]\ , \tag{3.9}$$

where $Q(j)$ is the quadratic Casimir operator of an irreducible representation of SU(3) which belongs to the class ψ_j and $E_0[j]$ is the value of the unknot in S^3 with preferred framing and with colour state ψ_j. Expression (3.9) has been obtained by taking into account the transformation property [1] of the expectation values under a modification of the framing of the link components.

If we denote by $\lambda_\pm$ the expectation value

$$\lambda_\pm=\langle \tilde{W}(U;\pm 1)\rangle|_{S^3}\ , \tag{3.10}$$

eq. (3.3) takes the form

$$\lambda_\pm \sum_j \xi(j)\,q^{\mp Q(j)}E_0[j] = \sum_j\sum_i \xi(j)\phi_\pm(i)\langle W(C;\psi_j)\,W(U;\psi_i)\rangle|_{S^3}\ , \tag{3.11}$$

where $\langle W(C;\psi_j)\,W(U;\psi_i)\rangle|_{S^3}=H_{ij}$ is the value of the Hopf link in S^3 with components U and C; both components U and C of the Hopf link, shown in Fig. 3.1, have preferred framings and are labelled by ψ_i and ψ_j, respectively.

The coefficients $\{\phi_\pm(i)\}$ must be chosen in such a way that eq. (3.3) holds for any link L. This means that eq. (3.11) must hold for arbitrary coefficients $\{\xi(j)\}$. Therefore, for any j, $\{\phi_\pm(i)\}$ must satisfy

$$\sum_i \phi_\pm(i)H_{ij}=\lambda_\pm\, q^{\mp Q(j)}E_0[j]\ . \tag{3.12}$$

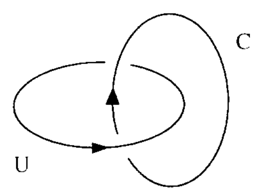


Fig. 3.1. Oriented Hopf link in S^3.

The values $\{H_{ij}\}$ of the Hopf link are interpreted as the matrix elements of a symmetric matrix H called the Hopf matrix. By definition of reduced tensor algebra, this matrix is non-singular and the linear system (3.12) determines the coefficients $\{\phi_\pm(i)\}$ uniquely up to a multiplicative constant. Before working out the general case, we shall illustrate how to solve eq. (3.12) in a few simple cases.

4. Elementary surgery operators for low values of *k*

When $k=1$, the reduced tensor algebra is of order three [1]; the elements of the standard basis of $\mathscr{T}_{(1)}$ are denoted by $\{\Psi[0], \Psi[1], \Psi[-1]\}$ where $\Psi[0]$ is the unit element, see Ref. [1]. In this case, eq. (3.12) takes the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \phi_\pm(0) \\ \phi_\pm(1) \\ \phi_\pm(-1) \end{pmatrix} = \lambda_\pm \begin{pmatrix} 1 \\ e^{\pm 2\pi i/3} \\ e^{\pm 2\pi i/3} \end{pmatrix}. \tag{4.1}$$

The solution of this system is

$$\begin{aligned} &\phi_\pm(0) = \pm(i/\sqrt{3})\lambda_\pm , \\ &\phi_\pm(-1) = \phi_\pm(1) = \pm(i/\sqrt{3})\lambda_\pm \, e^{\mp i2\pi/3} . \end{aligned} \tag{4.2}$$

The solution of eq. (4.1) depends linearly on the parameter $\lambda_\pm$; the (nonvanishing) value of $\lambda_\pm$ is free and the vacuum expectation values of the observables will not depend on it. For later convenience, we fix $\lambda_\pm$ to be

$$\lambda_\pm = e^{\mp i\pi/2} . \tag{4.3}$$

With this choice, the elementary surgery operators are

$$\tilde{W}(U; \pm 1) = W(U; \Psi_\pm) , \tag{4.4}$$

where U has preferred framing and

$$\Psi_\pm = (1/\sqrt{3})(\Psi[0] + e^{\mp i2\pi/3}\Psi[1] + e^{\mp i2\pi/3}\Psi[-1]) . \tag{4.5}$$

When $k=2$, the reduced tensor algebra $\mathscr{T}_{(2)}$ is isomorphic [1] with $\mathscr{T}_{(1)}$ and the Hopf matrix is the complex conjugate of the matrix shown in eq. (4.1). Consequently, one has

$$\begin{aligned} &\lambda_\pm = e^{\pm i\pi/2} , \\ &\Psi_\pm = (1/\sqrt{3})(\Psi[0] + e^{\pm i2\pi/3}\Psi[1] + e^{\pm i2\pi/3}\Psi[-1]) . \end{aligned} \tag{4.6}$$

For $k=3$, the reduced tensor algebra is of order one; the basis element $\Psi[0,0]$ can be represented by the element $\chi[0,0]$ of $\mathscr{T}$ corresponding to the trivial representation of SU(3). When $k=3$, surgery is realized in a trivial way because the elementary surgery operators coincide with the identity.

When $k=4$, the reduced tensor algebra $\mathscr{T}_{(4)}$ is of order three with basis elements

$$\psi_1 = \Psi[0,0] , \qquad \psi_2 = \Psi[1,0] , \qquad \psi_3 = \Psi[0,1] , \tag{4.7}$$

where $\{\Psi[0,0], \Psi[1,0], \Psi[0,1]\}$ correspond [1] to the three points of the fundamental domain Δ_4. Eq. (3.12) takes the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \phi_\pm(1) \\ \phi_\pm(2) \\ \phi_\pm(3) \end{pmatrix} = \lambda_\pm \begin{pmatrix} 1 \\ e^{\pm 2\pi i/3} \\ e^{\pm 2\pi i/3} \end{pmatrix} . \tag{4.8}$$

The solution of eq. (4.8) is

$$\begin{aligned} &\lambda_\pm = e^{\mp i\pi/2} , \\ &\Psi_\pm = (1/\sqrt{3})(\psi_1 + e^{\mp i2\pi/3}\psi_2 + e^{\mp i2\pi/3}\psi_3) . \end{aligned} \tag{4.9}$$

It is clear that, for every fixed value of k, the computation of the elementary surgery operators $\tilde{W}(U; \pm 1)$ is straightforward. Our main task now is to produce the general expression of $\tilde{W}(U; \pm 1)$ for any $k \geqslant 3$.

5. Elementary surgery operators for $k \geqslant 3$

In this section we shall determine the solution of system (3.12), which defines the elementary surgeries operators, for $k \geqslant 3$. Since the low values of k have been considered in the previous section, the results of this section completely determine the elementary surgery operators for all the permitted values of k.

Let us recall that each irreducible representation (m, n) of SU(3) corresponds to one element $\chi[m, n]$ of the tensor algebra $\mathscr{T}$ and can be represented by a point of coordinates (m, n) on a two-dimensional square lattice. In Ref. [1] we have shown that, for $k \geqslant 3$, the reduced tensor algebra $\mathscr{T}_{(k)}$ coincides with the classes of elements of $\mathscr{T}$ modulo the ideal $I_{(k)}$ generated by the vectors $\zeta_1 = \chi[k-2, 0]$ and $\zeta_2 = \chi[k-1, 0]$. The elements of the standard basis of $\mathscr{T}_{(k)}$ are $\{\Psi[a, b]\}$, where $\Psi[a, b]$ can be represented by $\chi[a, b]$ with $(a, b) \in \Delta_k$. The fundamental domain Δ_k of the square lattice is defined by

$$\Delta_k \equiv \{(a, b)\} \quad \text{with} \begin{cases} 0 \leqslant a < k-2 , \\ 0 \leqslant b < -a+k-2 . \end{cases} \tag{5.1}$$

From eq. (5.1) it follows that $\mathscr{T}_{(k)}$ is of order $d(k) = (k-1)(k-2)/2$. To simplify notations, we shall denote by $\{\psi_i\}$ the elements of the standard basis of $\mathscr{T}_{(k)}$ with $1 \leqslant i \leqslant d(k)$. The element ψ_1 corresponds to the unit $\Psi[0, 0]$ of the reduced tensor algebra. We can introduce the automorphism $*$ of $\mathscr{T}_{(k)}$ defined as

$$\Psi[n, m] \xrightarrow{*} \Psi[m, n] . \tag{5.2}$$

The $*$ transformation corresponds to the usual conjugation operation on the representations of SU(3). We shall denote $*$ by

$$\psi_i \xrightarrow{*} \psi_{i^*} . \tag{5.3}$$

The structure constants of $\mathscr{T}$ are given by

$$\chi[\rho_1]\chi[\rho_2] = \sum_\rho F_{\rho_1,\rho_2,\rho}\,\chi[\rho] , \tag{5.4}$$

where $F_{\rho_1,\rho_2,\rho}$ is the multiplicity of the irreducible representation ρ which is contained in the decomposition of the tensor product $\rho_1 \otimes \rho_2$. In the standard basis of $\mathscr{T}$, the structure constants take nonnegative integer values. By definition, one has [1]

$$F_{\rho_1,\rho_2,\rho} = F_{\rho_2,\rho_1,\rho} , \tag{5.5}$$

$$F_{\rho_1,\rho_2,\rho} = F_{\rho_1^*,\rho_2^*,\rho^*} , \tag{5.6}$$

$$F_{\rho_1,\rho_2,\rho} = F_{\rho_1^*,\rho,\rho_2} , \tag{5.7}$$

where ρ^* is the complex conjugate of the irreducible representation ρ.

The structure constants $\{N_{ijm}\}$ of $\mathscr{T}_{(k)}$ are determined by

$$\psi_i\psi_j = \sum_m N_{ijm}\psi_m . \tag{5.8}$$

Since $\mathscr{T}_{(k)} \simeq \mathscr{T}/I_{(k)}$, the structure constants $\{N_{ijm}\}$ are uniquely determined by $\{F_{\rho_1,\rho_2,\rho}\}$. Indeed, the decomposition of $\psi_i\psi_j$ can be obtained in two steps. Firstly, one has to analyze the corresponding decomposition in $\mathscr{T}$ of the product $\chi_i\chi_j$ of two representatives of ψ_i and ψ_j. Secondly, by using the correspondence rules given in Ref. [1], each element appearing the decomposition of $\chi_i\chi_j$ must be replaced by its corresponding class in $\mathscr{T}_{(k)}$. It is important to note that the correspondence rules between the elements of $\mathscr{T}$ and $\mathscr{T}_{(k)}$ introduce [1] well defined nontrivial signs which must be taken into account.

Since $\mathscr{T}_{(k)}$ is commutative and $*$ is an automorphism, one finds

$$N_{ijm} = N_{jim} , \qquad N_{ijm} = N_{i^*j^*m^*} . \tag{5.9}$$

Furthermore, as shown in Appendix A, one has

$$N_{ijm} = N_{i^*mj} . \tag{5.10}$$

From eqs. (5.9) and (5.10), it follows that

$$N_{ijm^*} = N_{imj^*} = N_{jmi^*} . \tag{5.11}$$

In order to construct the elementary surgery operators, we need to prove the following Lemma.

Lemma 1. *Let* $\mathscr{F}(i,j)$ *be a function on* $\mathscr{T}_{(k)} \otimes \mathscr{T}_{(k)}$. *Then, one has*

$$\sum_i \sum_j N_{imj} \mathscr{F}(i,j) = \sum_j \sum_i N_{jmi} \mathscr{F}(i^*, j^*) . \tag{5.12}$$

Proof. Since $*$ is an automorphism of $\mathscr{T}_{(k)}$ and the index j is summed over the whole algebra, the left-hand side of eq. (5.12) can be written as

$$\sum_i \sum_j N_{imj} \mathscr{F}(i,j) = \sum_i \sum_j N_{imj*} \mathscr{F}(i, j^*) . \tag{5.13}$$

By using eq. (5.11), one gets

$$\sum_i \sum_j N_{imj} \mathscr{F}(i,j) = \sum_i \sum_j N_{jmi*} \mathscr{F}(i, j^*) , \tag{5.14}$$

and, by using again the freedom to replace i with i^* in the sum, we have

$$\sum_i \sum_j N_{imj} \mathscr{F}(i,j) = \sum_j \sum_i N_{jmi} \mathscr{F}(i^*, j^*) ,$$

which coincides with eq. (5.12). □

Let us recall that the elementary surgery operators $\tilde{W}(U; \pm 1)$ are written as

$$\tilde{W}(U; \pm 1) = \sum_i \phi_\pm(i) W(U; \psi_i) , \tag{5.15}$$

where a given orientation has been introduced for the unknot U which has preferred framing. The coefficients $\{\phi_\pm(i)\}$ are fixed by the following theorem.

Theorem 1. *For $k \geqslant 3$, the elementary surgery operators are given by eq. (5.15) with*

$$\phi_\pm(i) = a(k) q^{\pm Q(i)} E_0[i] , \tag{5.16}$$

$$a(k) = \frac{16 \cos(\pi/k) \sin^3(\pi/k)}{k\sqrt{3}} . \tag{5.17}$$

Moreover, with the normalization choice (5.17), one has

$$\lambda_\pm = \langle \tilde{W}(U; \pm 1) \rangle |_{S^3} = e^{\pm i6\pi/k} . \tag{5.18}$$

Proof. We need to verify that the coefficients shown in eq. (5.16) satisfy eq. (3.12). The Hopf matrix H can be written [1] as

$$H_{ij} = q^{-Q(i)-Q(j)} \sum_m N_{ijm} E_0[m] q^{Q(m)} . \tag{5.19}$$

Therefore, if one inserts the values (5.16) for $\{\phi_+(i)\}$ in eq. (3.12), one finds

$$\lambda_+ q^{-Q(j)} E_0[j] = a(k) \sum_i \sum_m N_{ijm} q^{-Q(j)} q^{Q(m)} E_0[m] E_0[i] . \tag{5.20}$$

Since $E_0[i]=E_0[i^*]$ and $Q(i)=Q(i^*)$, by using Lemma 1 we have

$$\lambda_+ E_0[j]=a(k)\sum_i\sum_m N_{mji}q^{Q(m)}E_0[m]E_0[i] \ . \tag{5.21}$$

On the other hand, the values of the unknot give [1] a representation of the tensor algebra in $Z[q^{\pm 1/3}]$; for a fixed integer k, the values of the unknot give [1] a representation of $\mathscr{T}_{(k)}$ in $\mathbb{R}$. Consequently,

$$\sum_i N_{mji}E_0[i]=E_0[m]E_0[j] \ . \tag{5.22}$$

Thus, eq. (5.21) takes the form

$$\lambda_+ =a(k)\sum_i q^{Q(i)}E_0^2[i] \ . \tag{5.23}$$

It remains to be verified that eq. (5.23) is valid with the values (5.17) and (5.18) of $a(k)$ and λ_+. As shown in Appendix B, one has

$$Z_{(+1)}=\sum_i q^{Q(i)}E_0^2[i]=\frac{k\sqrt{3}}{16\cos(\pi/k)\,\sin^3(\pi/k)}\,\mathrm{e}^{\mathrm{i}6\pi/k} \ . \tag{5.24}$$

Therefore, eqs. (5.17) and (5.18) are in agreement with eq. (5.23). Clearly, the coefficients $\{\phi_-(i)\}$ and λ_- can be obtained by taking the complex conjugate of $\{\phi_+(i)\}$ and λ_+. □

Since the Hopf matrix is invertible [1], the coefficients $\{\phi_\pm(i)\}$ are uniquely determined by eq. (3.12) up to an overall numerical factor. In this sense, the solution (5.16) is unique. Our normalization choice (5.17) will simplify our subsequent discussion on Kirby moves but has no influence at all on the expectation values of the observables.

As we have mentioned in Section 2, each surgery link $\mathscr{L}$ is not oriented. On the other hand, in the definition (5.15) of the elementary surgery operators $\tilde{W}(U;\pm 1)$ we have introduced an orientation for the unknot U. Since $\tilde{W}(U;\pm 1)$ represent the elementary surgery operations $S_\pm$, the particular choice of this orientation must be irrelevant; let us now verify that this is really the case. Let us recall that, in general, if the oriented not C has colour ψ_i, a modification of the orientation of C is equivalent to replace ψ with ψ_{i^*}. Now, the coefficients $\{\phi_\pm(i)\}$ given in eq. (5.16) verify the equality $\phi_\pm(i)=\phi_\pm(i^*)$. Consequently, if we modify the orientation of the unknot U in eq. (5.15), the associated operators $\tilde{W}(U;\pm 1)$ are not modified. Thus, the particular choice of the orientation of U in eq. (5.15) is totally irrelevant; as it should be.

6. Surgery rules

In this section, we shall give the surgery rules in the quantum CS field theory with gauge group $G=\mathrm{SU}(3)$. We shall also discuss the invariance under Kirby moves of the results obtained according to these rules.

As we have shown in Ref. [1], for fixed integer k the physically inequivalent colour states associated with a knot are described by the elements of the reduced tensor algebra $\mathscr{T}_{(k)}$. In Ref. [1] we have also given, for each integer k, the correspondence rules between the elements of $\mathscr{T}$ and the elements of $\mathscr{T}_{(k)}$. Therefore, for fixed integer k, we only need to consider the complete set of observables consisting of Wilson line operators associated to framed oriented links whose components have colour states given by elements of $\mathscr{T}_{(k)}$. In the remaining part of this article, we shall concentrate on these observables.

Let us briefly summarize the results of Sections 4 and 5. The elementary surgery operators $\tilde{W}(U;\pm 1)$ are given by

$$\tilde{W}(U;\pm 1) = W(U;\Psi_{\pm}) , \tag{6.1}$$

where the unknot U has preferred framing and

$$\Psi_{\pm} = a(k) \sum_i q^{\pm Q(i)} E_0[i] \psi_i . \tag{6.2}$$

In eq. (6.2), the sum rules over all the elements of $\mathscr{T}_{(k)}$ and the values of the normalization factor $a(k)$ are

$$a(k) = \begin{cases} 1/\sqrt{3} & \text{for } k=1,2 , \\ (16/k\sqrt{3}) \cos(\pi/k) \sin^3(\pi/k) & \text{for } k \geqslant 3 . \end{cases} \tag{6.3}$$

Eq. (3.12) can be interpreted in the following way. The action of $\tilde{W}(U;\pm 1)$ on the element ψ_j of the reduced tensor algebra $\mathscr{T}_{(k)}$, associated with the core of the complement solid torus of U in S^3, is

$$\tilde{W}(U;\pm 1) : \psi_j \to e^{\mp i\varphi} q^{\mp Q(j)} \psi_j , \tag{6.4}$$

where the value of $e^{i\varphi}$ is

$$e^{i\varphi} = \begin{cases} e^{i\pi/2} & \text{for } k=1 , \\ e^{-i\pi/2} & \text{for } k=2 , \\ e^{-i6\pi/k} & \text{for } k \geqslant 3 . \end{cases} \tag{6.5}$$

Eq. (6.4) shows that, apart from the overall phase factor $e^{\pm i\varphi}$, $\tilde{W}(U;\pm 1)$ are the generators of twist homeomorphisms $\tau_{\mp}$ of the complement solid torus of U in S^3. Since these phase factors are constants (i.e., do not depend on the particular colour state under consideration), their presence in eq. (6.4) is completely harmless. In fact, by introducing the correct vacuum normalization in the expectation values, as shown in eq. (3.3), these phase factors cancel out.

By means of eqs. (6.1)–(6.5), we can now compute the expectation values of Wilson line operators for the CS theory defined in any orientable, closed connected three-manifold $\mathscr{M}$. We shall consider firstly the surgery rules in the case in which the surgery instructions have the form specified by the Fundamental Theorem. Then, we shall give the surgery rules corresponding to a generic "honest" surgery.

According to the Lickorish Theorem, $\mathscr{M}$ has a presentation in terms of Dehn surgery in S^3. Moreover, it is always possible to find a surgery description of $\mathscr{M}$ corresponding to a surgery link $\mathscr{L}$ in which all its components $\{\mathscr{L}_a\}$ (with $1 \leqslant a \leqslant p$) are simple circles and have surgery coefficients $\{r_a\}$ equal to $+1$ or -1.

Since we know how to represent in the field theory the elementary surgery $S_\pm$, associated with each single component in $\mathscr{L}$, we can construct the operator $\tilde{W}(\mathscr{L})$ representing the whole surgery. Indeed, for each component $\mathscr{L}_a$, we shall introduce a preferred framing $\mathscr{L}_{af}$ and consider the operator $\tilde{W}(\mathscr{L}_a; r_a)$, as defined in eq. (6.1). The field theory operator $\tilde{W}(\mathscr{L})$, associated to the whole surgery $\mathscr{L}$, is [3]

$$\tilde{W}(\mathscr{L}) = \prod_{a=1}^{p} \tilde{W}(\mathscr{L}_a; r_a) . \tag{6.6}$$

Let L be a given framed link in $\mathscr{M}$. As we have already mentioned, the isotopy class of $L \subset \mathscr{M}$ can be described by a link (that we indicate by the same symbol L) in the complement of $\mathscr{L}$ in S^3.

The expectation value $\langle W(L) \rangle |_{\mathscr{M}}$ is simply [3]

$$\langle W(L) \rangle |_{\mathscr{M}} = \langle W(L) \tilde{W}(\mathscr{L}) \rangle |_{S^3} / \langle \tilde{W}(\mathscr{L}) \rangle |_{S^3} . \tag{6.7}$$

By means of the surgery rules (6.6) and (6.7), one can compute the expectation values $\{\langle W(L) \rangle |_{\mathscr{M}}\}$ in any orientable, closed connected three-manifold $\mathscr{M}$. In order to discuss the invariance under Kirby moves, it is convenient to give the surgery rules which correspond to "honest" surgeries in general.

Any "honest" surgery is described by a surgery link $\mathscr{L}$ whose components $\{\mathscr{L}_a\}$ have integer surgery coefficients $\{r_a\}$. In this case, each single component $\mathscr{L}_a$ is not necessarily ambient isotopic with a simple circle, of course. For each component $\mathscr{L}_a$, we shall introduce a framing $\mathscr{L}_{af}$ such that the linking number of $\mathscr{L}_a$ and $\mathscr{L}_{af}$ satisfies

$$\mathrm{lk}(\mathscr{L}_a, \mathscr{L}_{af}) = r_a . \tag{6.8}$$

Then we shall consider the Wilson line operator

$$W(\mathscr{L}_a; \Psi_0) , \tag{6.9}$$

which is associated with the framed component $\mathscr{L}_a$ with framing $\mathscr{L}_{af}$ specified in eq. (6.8). The element Ψ_0 (surgery colour state) is given by

$$\Psi_0 = a(k) \sum_i E_0[i] \psi_i . \tag{6.10}$$

It should be noted that the framing choice (6.8) is different from the preferred framing convention which is used in the definition (6.1). Under a Kirby move, the integer surgery coefficient r_a, associated with a link component $\mathscr{L}_a$, transforms as the linking number $\mathrm{lk}(\mathscr{L}_a, \mathscr{L}_{af})$. For this reason, the framing choice (6.8) has an intrinsic meaning. The information carried by the integer surgery coefficient r_a is now encoded in the framing of the component $\mathscr{L}_a$ and, consequently, the colour state Ψ_0 of any surgery component is universal.

Theorem 2. *The surgery operator, corresponding to the "honest" surgery described by $\mathscr{L}=\{\mathscr{L}_a\}$ with integer surgery coefficients $\{r_a\}$ (with $1 \leqslant a \leqslant p$), is*

$$\tilde{W}(\mathscr{L}) = \prod_{a=1}^{p} W(\mathscr{L}_a; \Psi_0) , \tag{6.11}$$

where Ψ_0 is shown in eq. (6.10). The expectation value $\langle W(L) \rangle |_{\mathscr{M}}$ is given by

$$\langle W(L) \rangle |_{\mathscr{M}} = \langle W(L) \tilde{W}(\mathscr{L}) \rangle |_{S^3} / \langle \tilde{W}(\mathscr{L}) \rangle |_{S^3} . \tag{6.12}$$

The results obtained according to eq. (6.12) are invariant under Kirby moves.

Proof. By using the covariance properties [1] of the expectation values of the Wilson line operators under a change of framing, it is easy to verify that, when all the components $\{\mathscr{L}_a\}$ are simple circles and the surgery coefficients $\{r_a\}$ are equal to $+1$ or -1, eq. (6.12) coincides with eq. (6.7). In order to prove the consistency of eq. (6.12), we need to demonstrate that the results obtained according to eq. (6.12) are invariant under Kirby moves.

Let $\mathscr{L}$ be the instruction corresponding to a given "honest" surgery. Suppose that one component, say $\mathscr{L}_1$, of $\mathscr{L}$ is a simple circle with surgery coefficient $r_1 = \pm 1$. All the remaining components of $\mathscr{L}$ and the given link L belong to the complement solid torus $\mathscr{V}_1$ of $\mathscr{L}_1$ in S^3. As we have stated above, $W(\mathscr{L}_1; \Psi_0)$ is equivalent to $\tilde{W}(\mathscr{L}_1; \pm 1)$. Consider now the numerator and, separately, the denominator of the expression (6.12). From eq. (6.4) it follows that $\mathscr{L}_1$ can be eliminated provided that we perform a twist homeomorphism $\tau_{\mp}$ of $\mathscr{V}_1$ and, simultaneously, we multiply by the phase factor $e^{\mp i\varphi}$. This phase factor cancels out in the ratio (6.12); therefore, only the effects of the twist homeomorphism $\tau_{\mp}$ are relevant. Under this twist, L is accordingly modified into its homeomorphic image, as it should be. The same happens with the remaining components of $\mathscr{L}$. It remains to be verified that the new surgery coefficients $\{r'_b\}$ (with $b \neq 1$) have the correct values shown in eq. (2.5); as a consequence of the transformation properties [2] of framings under twist homeomorphisms, this is indeed the case.

In conclusion, the results obtained by means of the surgery rules (6.11) and (6.12) are invariant under Kirby moves. On the other hand, with an appropriate sequence of Kirby moves, any "honest" surgery instruction can be transformed into an equivalent surgery instruction of the type specified by the Fundamental Theorem. In this case, expressions (6.12) and (6.7) coincide and this concludes the proof. □

With our definition of the elements $\Psi_\pm$ shown in eq. (6.2), Ψ_- can be obtained from Ψ_+ by means of two elementary changes of framing. This is the main reason of our choice on their normalization.

From the previous discussion on the Kirby moves, it is clear that the presence of the phase factors $e^{\pm i\varphi}$ cannot be avoided. To be more precise, whatever the normalization of $\Psi_\pm$ is, the numerator (and, separately, the denominator) of the expression (6.12) is not invariant under Kirby moves. (The ratio (6.12) is invariant under Kirby moves, of course.) In other words, a projective representation of the group of twist homeomorphisms of solid tori is realized on the state space of the CS theory.

When $\langle \tilde{W}(\mathscr{L}) \rangle|_{S^3} \neq 0$ the expression (6.12) is well defined. The expectation value $\langle \tilde{W}(\mathscr{L}) \rangle|_{S^3}$ gives information on the three-manifold $\mathscr{M}$, associated with the surgery link $\mathscr{L}$ in S^3, and depends on the value of the coupling constant k. For fixed $\mathscr{M}$, $\langle \tilde{W}(\mathscr{L}) \rangle|_{S^3}$ may vanish when k takes values on a certain set of integers; in this case, eq. (6.12) is not well defined. Let us recall that, in general, the internal consistency of the quantum CS theory defined in $\mathscr{M}$ puts some restrictions [6] on the possible values of k. It is natural to expect that $\langle \tilde{W}(\mathscr{L}) \rangle|_{S^3}$ vanishes for precisely those values of k for which the quantum CS theory is not well defined in $\mathscr{M}$.

7. General properties

The derivation of the field theory surgery rules for SU(3) that we have presented in the previous sections is based on the general arguments described in Ref. [3] for the case of SU(2). Even if the numerical results are different for different gauge groups, the underlying algebraic structure is universal. It is remarkable that this universal structure is not peculiar of topological quantum field theory but appears also in all the different approaches [7–12] to the new discovered three-manifold invariants. Thus, before considering explicit applications of the surgery rules in the SU(3) CS field theory, we would like to present here some general results concerning the construction of Dehn surgery operators.

Let us consider the CS field theory with compact simple Lie group G. For generic values of the coupling constant k, the expectation values of the Wilson line operators are finite Laurent polynomials in a certain power of the deformation parameter $q = \exp(-i2\pi/k)$. By construction, for any fixed link with n components, these expectation values are multilinear functions on $\mathscr{T}^{\otimes n}$, where $\mathscr{T}$ is the tensor algebra (or complexification of the representation ring) of G. Invariance under large gauge transformations in S^3 implies that k is integer [6]. For fixed integer k, the kernel in $\mathscr{T}$ which is determined by the expectation values is an ideal [1] denoted by $I_{(k)}$. Consequently, for fixed integer k, the space of the physically inequivalent colour states associated with a knot is simply given by the classes of elements of $\mathcal{T}$ modulo the ideal $I_{(k)}$; these classes form an algebra which is called the reduced tensor algebra $\mathcal{T}_{(k)}$. Thus, (for fixed integer k) a complete set of gauge invariant observables is represented by the set of Wilson line operators associated with framed oriented links whose components have colour states given by elements of $\mathcal{T}_{(k)}$. In the considered examples where G is a unitary group, $\mathcal{T}_{(k)}$ turns out to be of finite order. It is natural to expect that $\mathcal{T}_{(k)}$ is of finite order also for a generic (simple compact Lie) group G. In what follows, we simply assume that this is indeed the case.

For generic values of k, the generalized satellite relations [1] are written in terms of the elements of $\mathcal{T}$. For fixed integer k, these relations can be expressed in terms of the elements $\{\psi_i\}$ of the standard basis of $\mathcal{T}_{(k)}$. For example, consider the knots C_1 and C_2 in the solid torus V which coincides with the complement of the unknot U in S^3, as shown in Fig. 7.1. Let these knots have preferred framings and colours ψ_i and ψ_j belonging to $\mathcal{T}_{(k)}$. The product $W(C_1\psi_i)W(C_2;\psi_j)$ of the associated Wilson line operators admits the decomposition

$$W(C_1;\psi_i)W(C_2;\psi_j) = \sum_m N_{ijm} W(C_1;\psi_m) , \tag{7.1}$$

where $\{N_{ijm}\}$ are the structure constants of $\mathcal{T}_{(k)}$.

At this point, one should construct the surgery operators $\tilde{W}(U;\pm 1)$ which are determined by eqs. (3.6) and (3.12). Clearly, the Hopf matrix H entering eq. (3.12) depends on G; nevertheless, we shall show that the solution of eq. (3.12), namely the form of $\tilde{W}(U;\pm 1)$, does not depend on G. Our proof is based exclusively on the topological properties of surgery; in particular, we will not use the structure of the Hopf matrix.

Let us consider the set of "honest" surgeries. Each surgery of this kind can be described by a framed unoriented link $\mathcal{L}=\{\mathcal{L}_a\}$ in S^3; we use the standard convention in which each surgery coefficient r_a is given by the linking number of the component $\mathcal{L}_a$ and its framing $\mathcal{L}_{af}$, eq. (6.8). As we have already mentioned, with the choice (6.8) of framing, the surgery operators are characterized by a universal colour state Ψ_0, see eq. (6.9). The statement that the form of $\tilde{W}(U;\pm 1)$ does not depend on G is equivalent to the statement that Ψ_0 has the structure given in eq. (6.10).

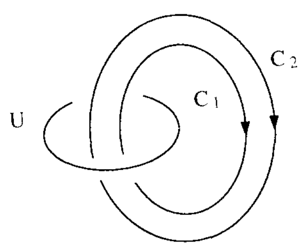


Fig. 7.1. Knots C_1 and C_2 in the complement of U in S^3.

Definition. Let $\mathscr{T}_{(k)}$ be the reduced tensor algebra in the Chern–Simons theory with compact simple Lie gauge group G. We say that $\mathscr{T}_{(k)}$ is regular if the following properties are satisfied:

(i) $\mathscr{T}_{(k)}$ is of finite order;

(ii) there is a standard basis of elements $\{\psi_i\}$ in $\mathscr{T}_{(k)}$ such that the structure constants $\{N_{ijm}\}$, defined by

$$\psi_i \psi_j = \sum_k N_{ijm} \psi_m ,$$

satisfy the relations (5.9)–(5.10);

(iii) with respect to the standard basis defined above, one has

$$\lambda_+ = a(k) \sum_i q^{Q(i)} E_0^2[i] \neq 0 ,$$

where $E_0[i]$ is the value of the unknot, which has preferred framing and colour $\psi_i \in \mathscr{T}_{(k)}$.

Theorem 3. *Let us assume that the reduced tensor algebra $\mathscr{T}_{(k)}$ in the Chern–Simons theory with compact simple Lie gauge group G is regular. The surgery operator $\tilde{W}(\mathscr{L})$, which is associated to the surgery link $\mathscr{L} = \{\mathscr{L}_a\}$ with integer surgery coefficients $\{r_a\}$ and framings specified by eq. (6.8), is given by eq. (6.11) with*

$$\Psi_0 = a(k) \sum_i E_0[i] \psi_i , \tag{7.2}$$

where $a(k)$ is a nonvanishing normalization factor.

Proof. First of all we note that, in the computation of the expectation values (6.12), the particular value of the normalization factor $a(k)$ is irrelevant. (The natural choice for the value of $a(k)$ will be described in a while.) The colour state Ψ_0, given in eq. (7.2), coincides with the surgery colour state found for SU(2) [3] and SU(3). Thus, in these cases, the statement of Theorem 3 is true. Let us now consider a generic group G.

Since the element $\Psi_0 \in \mathscr{T}_{(k)}$ does not depend on the particular form of the surgery link $\mathscr{L} \in S^3$, Ψ_0 must be determined by general topological properties of surgery. In order to describe these properties, we need to disentangle the action of a generic surgery operation, which is defined inside a solid torus V, from the particular embeddings of this solid torus in S^3. To be more precise, let us represent the solid torus V by the complement of the unknot U in S^3; we shall denote by K the framed core of V with preferred framing. Suppose that K represents the surgery instruction corresponding to the "honest" Dehn surgery with surgery coefficient $r=0$. Then, each framed component $\mathscr{L}_a$ of a surgery link $\mathscr{L}$ in S^3 can be understood to be the image $h^{\diamond}(K)$ of K under the homeomorphism $h^{\diamond}$ which has been defined in Ref. [1].

Since the element Ψ_0 does not depend on $h^\diamond$, in order to find Ψ_0 we only need to consider the surgery operation described by K in V. By definition, this surgery consists of removing a tubular neighbourhood N of K in V and sewing it back in such a way that a meridian of N is mapped into the curve Y which is shown in Fig. 7.2. Since a meridian of N bounds a disc (standardly embedded in a three-ball), the knot Y shown in Fig. 7.2 is really ambient isotopic with the unknot (simple circle) contained inside a three-ball. Clearly, if the knot Y is framed, then Y is ambient isotopic with the unknot with preferred framing. This is the desired property which characterizes completely the surgery operation described by K in V.

We would now like to represent this surgery by a Wilson line operator $W(K; \Psi_0)$ which is defined for K with preferred framing. As we have already mentioned, a given orientation for K is also introduced, but the final results will not depend on the choice of this orientation. Suppose that a Wilson line operator $W(Y; \psi_j)$ is associated with the knot Y which is oriented and has preferred framing. Since Y is ambient isotopic with the unknot (contained inside a three-ball) with preferred framing, the Wilson line operator $W(Y; \psi_j)$ simply gives [3] the contribution $E_0[j]$. Therefore, the element Ψ_0 must be determined in such a way that, inside the solid torus V, the insertion of $W(Y; \psi_j)$ (with arbitrary ψ_j) is equivalent to the multiplication by $E_0[j]$. We shall now verify that the element Ψ_0, given in eq. (7.2), has precisely this property.

Let us consider the product $W(K; \Psi_0)\,W(Y; \psi_j)$ of the two Wilson line operators associated with K and Y inside the solid torus V. By using the decomposition (7.1) one finds

$$W(K; \Psi_0)\,W(Y; \psi_j) = a(k) \sum_i E_0[i] \sum_m N_{ijm}\, W(K; \psi_m) \;, \tag{7.3}$$

and, by means of Lemma 1, one obtains

$$W(K; \Psi_0)\,W(Y; \psi_j) = a(k) \sum_i E_0[i^*] \sum_m N_{mji}\, W(K; \psi_{m^*}) \;. \tag{7.4}$$

Since the values of the unknot give a representation of $\mathscr{T}_{(k)}$, the relation (5.22) is valid; moreover, $E_0[i^*] = E_0[i]$. Therefore, eq. (7.4) takes the form

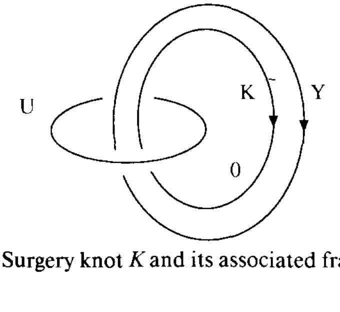


Fig. 7.2. Surgery knot K and its associated framing Y.

$$W(K;\Psi_0)\,W(Y;\psi_j)=E_0[j]a(k)\sum_m E_0[m]\,W(K;\psi_m)$$
$$=E_0[j]\,W(K;\Psi_0)\,. \tag{7.5}$$

This equation holds for any $\psi_j\in\mathcal{T}_{(k)}$. Therefore, eq. (7.5) shows that the operator $W(K;\Psi_0)$, with Ψ_0 given in eq. (7.2), represents the surgery operation which is described by the surgery knot K with surgery coefficient $r=0$.

In order to prove the invariance under Kirby moves, one has to consider the elementary surgery operators $\tilde{W}(U;\pm1)$. The operators $\tilde{W}(U;\pm1)$ can be obtained from $W(K;\Psi_0)$ by using the satellite relations. Let us consider a satellite of the unknot, with writhe $+1$ in S^3, which has been obtained by means of the pattern link defined by the two knots K and Y in V. This satellite is shown in Fig. 7.3. According to eq. (7.5), the expectation value of the Wilson line operators associated to the link of Fig. 7.3 is equal to $E_0[j]\langle W(U;\Psi_+)\rangle|_{S^3}$. Therefore, by taking into account the behaviour of the expectation values under a change of framing, one obtains

$$a(k)\sum_i q^{Q(i)}E_0[i]H_{ij}=q^{-Q(j)}E_0[j]\left(a(k)\sum_i q^{Q(i)}E_0^2[i]\right). \tag{7.6}$$

This equation coincides with eq. (3.12) with

$$\phi_+(i)=a(k)q^{Q(i)}E_0[i]\,, \tag{7.7}$$

$$\lambda_+=a(k)\sum_i q^{Q(i)}E_0^2[i]\,. \tag{7.8}$$

Clearly, ϕ_- and λ_- can be obtained from ϕ_+ and λ_+ taking the complex conjugates. By assumption, $\lambda_+\neq0$ and consequently $\lambda_-\neq0$. Therefore, the invariance under Kirby moves can be proved by using the same argument presented in Section 6. The natural choice of the normalization factor $a(k)$ is to require that $a(k)>0$ and $|\lambda_\pm|=1$. This is the convention that we have adopted in the previous sections. □

By definition of reduced tensor algebra, $\mathcal{T}_{(k)}$ is physically irreducible [1]; this means that if $\psi\in\mathcal{T}_{(k)}$ is physically equivalent to the null vector, then $\psi=0$. It should be noted that, in the proof of Theorem 3, this property of $\mathcal{T}_{(k)}$ has not been

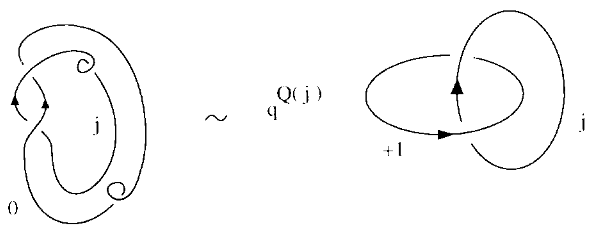


Fig. 7.3. Satellite of the unknot with writhe +1.

used. Theorem 3 provides an explicit solution to the problem of the surgery colour state Ψ_0; the uniqueness of this solution follows from Theorem 4.

Theorem 4. *Up to a normalization factor, the surgery colour state Ψ_0 is unique.*

Proof. Suppose that we have found two surgery colour states Ψ_0 and Ψ'_0 in $\mathcal{T}_{(k)}$. Let Ψ_+ and Ψ'_+ be the corresponding colour states associated with the elementary surgery S_+. Let us fix the normalization of Ψ_+ and Ψ'_+ (and, consequently, of Ψ_0 and Ψ'_0) by imposing that eq. (3.12) takes the form

$$(H\cdot\Psi_+)_i = q^{-Q(i)} E_0[i] = (H\cdot\Psi'_+)_i \,. \tag{7.9}$$

With this normalization, the colour state $\Psi = \Psi_+ - \Psi'_+$ satisfies

$$H\cdot\Psi = 0 \,. \tag{7.10}$$

Eq. (7.10) implies that, for any link $L \subset S^3$ in which one of its components C has colour Ψ, one has $\langle W(L)\rangle|_{S^3} = 0$. Indeed, by using the surgery operators, one can find a surgery presentation [2] of $L \subset S^3$ in which C is the unknot with colour Ψ. Of course, the remaining components of L and the surgery link belong to the complement solid torus of C in S^3. Therefore, by using the generalized satellite relations [1], $\langle W(L)\rangle|_{S^3}$ can be expressed in terms of the values of the Hopf link in which one of its components has colour Ψ. Consequently, eq. (7.10) implies that $\langle W(L)\rangle|_{S^3} = 0$.

On the other hand, since $\mathcal{T}_{(k)}$ is physically irreducible, one has $\Psi = 0$ and thus $\Psi_+ = \Psi'_+$; this implies that $\Psi_0 = \Psi'_0$. □

The existence of an operator which represents surgery (Theorem 3) implies that the Hopf matrix H_{ij} is nonsingular. Indeed, suppose that eq. (7.10) is satisfied with a certain colour state Ψ. By using the method described in the proof of Theorem 4, one can conclude that either $\mathcal{T}_{(k)}$ is physically reducible or $\Psi = 0$. Since the reduced tensor algebra $\mathcal{T}_{(k)}$ is (by definition) physically irreducible, it must be $\Psi = 0$; this means that H is nonsingular.

8. The manifold $S^2\times S^1$

The manifold $S^2\times S^1$ admits [2] a surgery presentation in which the surgery link is the unknot U with surgery coefficient $r = 0$. The manifold $S^2\times S^1$ can also be represented by the region of $\mathbb{R}^3$ delimited by two spherical surfaces which are centred at the origin and have different radii; the points which have the same angular coordinates on the two spheres are identified.

The simplest knot C which is homotopically nontrivial in $S^2\times S^1$ is shown in Fig. 8.1. The link in S^3 shown in Fig. 8.1a has two components; one component

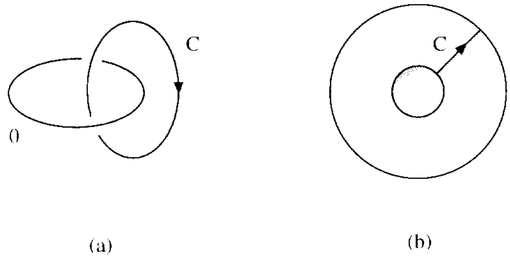


Fig. 8.1. Knot C in $S^2 \times S^1$.

represents the surgery instruction corresponding to the Dehn surgery on S^3 which gives $S^2 \times S^1$. The remaining component, which has preferred framing, describes the knot C. Fig. 8.1b gives an equivalent description of C in $S^2 \times S^1$.

Let C have colour ψ_i; the expectation value of the associated Wilson line operator in $S^2 \times S^1$ is given by Theorem 2,

$$\langle W(C; \psi_i) \rangle|_{S^2 \times S^1} = \langle W(C; \psi_i) \tilde{W}(U; \Psi_0) \rangle|_{S^3} / \langle \tilde{W}(U; \Psi_0) \rangle|_{S^3} . \tag{8.1}$$

On the one hand, one has

$$\langle W(C; \psi_i) \tilde{W}(U; \Psi_0) \rangle|_{S^3} = a(k) \sum_j E_0[j] H_{ji} . \tag{8.2}$$

On the other hand, one gets

$$\langle \tilde{W}(U; \Psi_0) \rangle|_{S^3} = a(k) \sum_j E_0^2[j] . \tag{8.3}$$

As shown in Appendix B, one finds

$$\sum_j E_0^2[j] = \frac{3k^2}{256 \sin^6(\pi/k) \cos^2(\pi/k)} , \tag{8.4}$$

$$\sum_j E_0[j] H_{ji} = \delta_{1i} \left(\frac{3k^2}{256 \sin^6(\pi/k) \cos^2(\pi/k)} \right) , \tag{8.5}$$

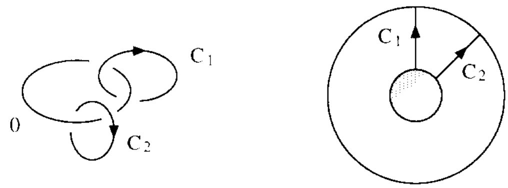
where ψ_1 denotes the unit element in $\mathscr{T}_{(k)}$; let us recall that the unit element of $\mathscr{T}_{(k)}$ was indicated by $\Psi[0]$ for $k=1$ and $k=2$, and by $\Psi[0, 0]$ for $k \geqslant 3$. From eqs. (8.4) and (8.5) it follows that

$$\langle W(C; \psi_i) \rangle|_{S^2 \times S^1} = \delta_{1i} . \tag{8.6}$$

Let us now consider the two components link shown in Fig. 8.2; the two components C_1 and C_2 shown in Fig. 8.2a have preferred framings and colours ψ_i and ψ_j, respectively. An equivalent description of C_1 and C_2 in $S^2 \times S^1$ is given in Fig. 8.2b. By using the satellite formulae [1], from eq. (8.6) one obtains

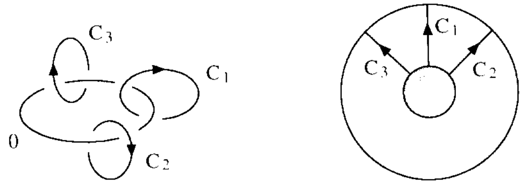
$$\langle W(C_1, C_2; \psi_i, \psi_j) \rangle|_{S^2 \times S^1} = N_{ij1} = \delta_{ij*} . \tag{8.7}$$

The three components of the link shown in Fig. 8.3 have preferred framings



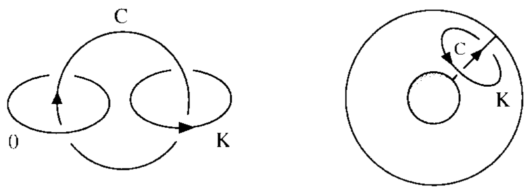
(a) (b)

Fig. 8.2. Knots C_1 and C_2 in $S^2 \times S^1$.



(a) (b)

Fig. 8.3. Three-component link in $S^2 \times S^1$.



(a) (b)

Fig. 8.4. Knots C and K in $S^2 \times S^1$.

and colours ψ_i, ψ_j and ψ_m; by using the satellite relations one finds

$$\langle W(C_1, C_2, C_3; \psi_i, \psi_j, \psi_m) \rangle |_{S^2 \times S^1} = \sum_n N_{ijn} \delta_{nm^*} = N_{ijm^*} . \tag{8.8}$$

The expectation value of the three-component link shown in Fig. 8.3 provides a direct representation of the structure constants $\{N_{ijm^*}\}$ of the reduced tensor algebra $\mathscr{T}_{(k)}$. Eq. (8.8) shows that N_{ijm^*} is symmetric under a generic permutation of the indices; this is in agreement with eq. (5.11).

In order to verify the ambient isotopy invariance of the expectation values in $S^2 \times S^1$, let us consider for example the link shown in Fig. 8.4a; the components C and K have preferred framing and colours ψ_i and ψ_j, respectively. On the one hand, according to the surgery rule (6.12), one obtains

$$\langle W(C; \psi_i) W(K; \psi_j) \rangle |_{S^2 \times S^1} = E_0[j] \delta_{i1} . \tag{8.9}$$

On the other hand, by means of an isotopy in $S^2 \times S^1$ (see Fig. 8.4b) one can move the knot K inside a three-ball. Consequently, $W(K;\psi_j)$ gives the contribution $E_0[j]$; thus

$$\langle W(C;\psi_i)\,W(K;\psi_j)\rangle|_{S^2\times S^1} = E_0[j]\,\langle W(C,;\psi_i)\rangle|_{S^2\times S^1}$$
$$= E_0[j]\delta_{i1}\,, \qquad (8.10)$$

in agreement with eq. (8.9).

The two components C_1 and C_2 of the link shown in Fig. 8.5 have preferred framings and colours ψ_i and ψ_j, respectively. By using the surgery rules, one can compute the expectation value of the associated Wilson line operators; one gets

$$\langle W(C_1;\psi_i)\,W(C_2;\psi_j)\rangle|_{S^2\times S^1} = q^{-2Q(j)}\delta_{ij*}\,. \qquad (8.11)$$

The result (8.11) can also be derived by using of the invariance under Kirby moves. Indeed, the sequence of Kirby moves shown in Fig. 8.6 and eq. (8.7) imply

$$\langle W(C_1;\psi_i)\,W(C_2;\psi_j)\rangle|_{S^2\times S^1} = q^{-Q(i)-Q(j)}\delta_{ij*}\,. \qquad (8.12)$$

The vacuum expectation values of the Wilson line operators in $S^2\times S^1$ can be obtained by means of a simple general rule. Let us consider the surgery presentation of $S^2\times S^1$ given by the unknot U in S^3 with surgery coefficient $r=0$. A generic link in $S^2\times S^1$ can be represented by a link L in the complement solid torus N of U in S^3. Within the solid torus N, the associated Wilson line operator $W(L)$ admits the decomposition

$$W(L) = \sum_i \xi_L(i)\,W(C;\psi_i)\,, \qquad (8.13)$$

where C is the oriented and framed core of N shown in Fig. 8.1a. Now, by using

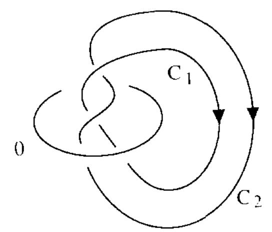


Fig. 8.5. Knots C_1 and C_2 in $S^2\times S^1$

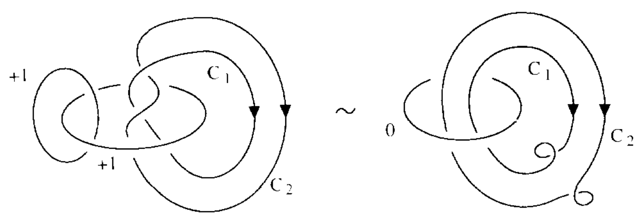


Fig. 8.6. Equivalent description of the link shown in Fig. 8.5.

eq. (8.6), one finds

$$\langle W(L) \rangle |_{S^2 \times S^1} = \xi_L(1) \, . \tag{8.14}$$

Since, for any link L, the ξ-coefficients can be determined uniquely (for example, by means of a Hopf matrix), eq. (8.14) gives a compact description of $\langle W(L) \rangle |_{S^2 \times S^1}$.

9. Lens spaces and Poincaré manifold

The lens space $L(p, 1)$ admits [2] a surgery presentation described by the unknot U with surgery coefficient $r=p$. In order to illustrate the use of the surgery rules, let us consider the case in which $k=1$. The simplest nontrivial knot C in $L(p, 1)$ is shown in Fig. 9.1; let C have preferred framing and colour ψ_i. Let us recall that $\mathscr{T}_{(3)}$ is of order three and the corresponding Hopf matrix is shown in eq. (4.1). From the definition of surgery operator, one has

$$\langle W(C; \psi_i) \rangle |_{L(p,1)} = \frac{\sum_j q^{pQ(j)} E_0[j] H_{ji}}{\sum_j q^{pQ(j)} E_0^2[j]} \, , \tag{9.1}$$

$$\langle W(C; \psi_i) \rangle |_{L(p,1)} = \begin{cases} 1 & \text{if } \psi_i = \Psi[0] \, , \\ (1 - \mathrm{e}^{-\mathrm{i}2\pi p/3})(1 + 2\mathrm{e}^{-\mathrm{i}2\pi p/3})^{-1} & \text{if } \psi_i = \Psi[\pm 1] \, . \end{cases} \tag{9.2}$$

When $k=2$, $\langle W(C; \psi_i) \rangle |_{L(p,1)}$ can be obtained by taking the complex conjugate on the expression (9.2). As we have already mentioned, the case $k=3$ is trivial. For $k=4$, one gets

$$\langle W(C; \psi_i) \rangle |_{L(p,1)} = \begin{cases} 1 & \text{if } \psi_i = \Psi[0] \, , \\ (1 - e^{-\mathrm{i}2\pi p/3})(1 + 2e^{-\mathrm{i}2\pi p/3})^{-1} & \text{if } \psi_i = \Psi[1, 0] \, , \\ (1 - e^{-\mathrm{i}2\pi p/3})(1 + 2e^{-\mathrm{i}2\pi p/3})^{-1} & \text{if } \psi_i = \Psi[0, 1] \, . \end{cases} \tag{9.3}$$

The general expression of $\langle W(L) \rangle |_{L(p,1)}$ can be obtained by using the decomposition (8.13) for $W(L)$, where L belongs to the complement solid torus of U in S^3. Similarly to the case of the manifold $S^2 \times S^1$, we only need to consider the knot C shown in Fig. 9.1 with preferred framing and colour state ψ_i. Let us consider first $\langle W(L) W(U, \Psi_0) \rangle |_{S^3}$, where the unknot U has framing specified by eq. (6.8) with $r=p$. By means of two Kirby moves, the link shown in Fig. 9.1 can be transformed as shown in Fig. 9.2. Therefore, by using eq. (8.8) and by taking into account the normalization factor, one finds

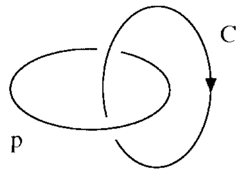


Fig. 9.1. Knot C in $L(p, 1)$.

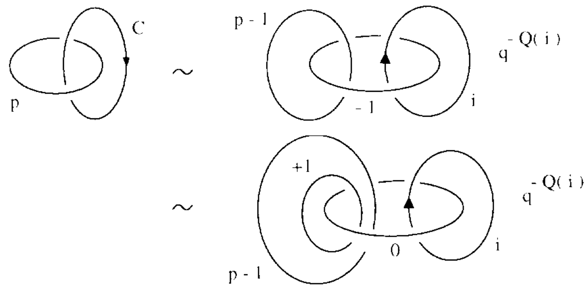


Fig. 9.2. Equivalent descriptions of the knot C in $L(p, 1)$.

$$\langle W(C;\psi_i) W(U,\Psi_0)\rangle|_{S^3} = a(k) \sum_{jm} N_{jmi*} E_0[j] E_0[m] q^{Q(j)} q^{(p-1)Q(m)} q^{-Q(i)} . \tag{9.4}$$

Let us denote by $Z_{(p)}$ the expectation value

$$Z_{(p)} = \langle W(U,\Psi_0)\rangle|_{S^3} = a(k) \sum_i q^{pQ(i)} E_0^2[i] . \tag{9.5}$$

For $Z_{(p)} \neq 0$, the expectation value $\langle W(L)\rangle|_{L(p,1)}$ is given by

$$\langle W(L)\rangle|_{L(p,1)} = Z_{(p)}^{-1} \sum_{jmi} \xi_L(i) N_{jmi*} E_0[j] E_0[m] \times q^{Q(j)} q^{(p-1)Q(m)} q^{-Q(i)} . \tag{9.6}$$

The last example of this section is the Poincaré manifold $\mathscr{P}$. This manifold is a homology sphere but not a homotopy sphere. A surgery presentation of $\mathscr{P}$ is given by the right-handed trefoil knot T in S^3 with surgery coefficient $r=1$ (and framing specified by eq. (6.8)). The knot $C \subset \mathscr{P}$ shown in Fig. 9.3 has preferred framing. For simplicity, we shall concentrate on the case $k=1$. By using the result (13.9) of Ref. [1], one obtains

$$\langle W(C;\psi_i) W(T,\Psi_0)\rangle|_{S^3} = \begin{cases} -\mathrm{i} & \text{for } \psi_i = \Psi[0] , \\ (\sqrt{3}+\mathrm{i})/2 & \text{for } \psi_i = \Psi[\pm 1] . \end{cases} \tag{9.7}$$

Therefore,

$$\langle W(C;\psi_i)\rangle|_{\mathscr{P}} = \begin{cases} 1 & \text{for } \psi_i = \Psi[0] , \\ (\mathrm{i}\sqrt{3}-1)/2 & \text{for } \psi_i = \Psi[\pm 1] . \end{cases} \tag{9.8}$$

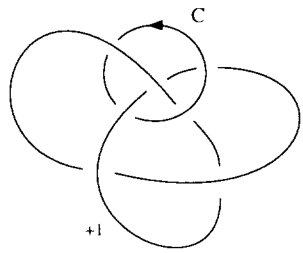


Fig. 9.3. Knot C in the Poincaré manifold $\mathscr{P}$.

It is clear that the corresponding expressions for $k=2$ can be obtained from (9.8) by taking the complex conjugate and the results for $k=4$ coincide with (9.8). For higher values of k, the computation of $\langle W(L) \rangle |_{\mathscr{P}}$ is straightforward.

10. Manifolds $\Sigma_g \times S^1$

In this section we shall consider the set of manifolds $\Sigma_g \times S^1$, where Σ_g is a Riemann surface of genus g. Let us denote by $\mathscr{L}(g)$ a surgery link in S^3 corresponding to a surgery presentation of $\Sigma_g \times S^1$. We shall describe firstly how to construct $\mathscr{L}(g)$ for arbitrary g. Then, we shall consider examples of links in $\Sigma_g \times S^1$.

The manifold Σ_g can be obtained from the two-sphere S^2 by adding g handles, of course. Similarly, the three-manifold $\Sigma_g \times S^1$ can be obtained from $S^2 \times S^1$ by "adding g handles" according to the prescription described in Ref. [13]. Consider the link $L(g)$ in S^3 shown in Fig. 10.1, where the component U and the g components $\{M_1, ..., M_g\}$ have preferred framings. The surgery link $\mathscr{L}(g)$ is a satellite of $L(g)$ which is obtained by replacing each component M_i (with $1 \leqslant i \leqslant g$) with $h^{\diamond}(P)$. The homeomorphism $h^{\diamond}$ has been defined in Ref. [1]. The pattern link P, which is contained in the complement solid torus N of the unknot V in S^3, is shown in Fig. 10.2. The satellite obtained according to this prescription is the link $\mathscr{L}(g)$ in S^3 which has $2g+1$ components. Each component of $\mathscr{L}(g)$ has preferred framing and, consequently, the associated surgery coefficient is $r=0$.

As an example, the surgery link $\mathscr{L}(1)$ is shown in Fig. 10.3; $\mathscr{L}(1)$ is ambient isotopic with the Borromean rings and corresponds to the manifold

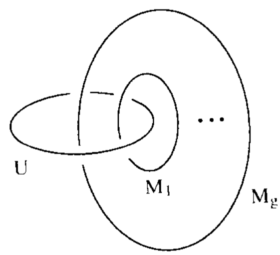


Fig. 10.1. Link $L(g)$ in S^3.

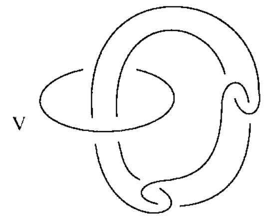


Fig. 10.2. Pattern link P in the complement of V in S^3.

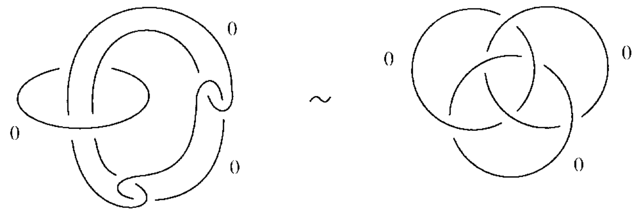


Fig. 10.3. Borromean rings.

$\Sigma_g \times S^1 \equiv T^2 \times S^1$. Suppose that both components of the pattern link P have colour Ψ_0. We shall denote by $W(P;\, \Psi_0,\, \Psi_0)$ the product of the associated Wilson line operators. Since P is defined in N, $W(P;\, \Psi_0,\, \Psi_0)$ is a gauge invariant observable defined inside a solid torus. Consequently it admits a decomposition of the type [1]

$$W(P;\, \Psi_0,\, \Psi_0) = \sum_i \eta(i)\, W(K;\, \psi_i)\ , \tag{10.1}$$

where K is the core of N with preferred framing. By using the method described in Ref. [13], one finds

$$\eta(i) = \sum_j E_0^{-1}[j] H_{ji}\ . \tag{10.2}$$

Eq. (10.2) is valid for any group G. Eq. (10.1) gives the decomposition of the surgery operator, which is associated to "one handle", in terms of a single coloured link component. The above decomposition will be useful in the computation of expectation values of Wilson line operators in $\Sigma_g \times S^1$.

Let consider for example the case $k=5$; the reduced tensor algebra $\mathscr{T}_{(5)}$ is of order six and the elements of its standard basis are

$$\{\Psi[0,0],\, \Psi[1,0],\, \Psi[2,0],\, \Psi[0,1],\, \Psi[0,2],\, \Psi[1,1]\}\ . \tag{10.3}$$

The nontrivial structure constants are given by

$$\Psi[1,0]\Psi[1,0] = \Psi[2,0] + \Psi[0,1]\ ,$$

$$\Psi[1,0]\Psi[0,1] = \Psi[0,0] + \Psi[1,1]\ ,$$

$$\Psi[1,0]\Psi[2,0] = \Psi[1,1]\ ,$$

$$
\begin{aligned}
&\Psi[1,0]\Psi[0,2]=\Psi[0,1]+\Psi[1,1]\,,\\
&\Psi[1,0]\Psi[1,1]=\Psi[1,0]+\Psi[0,2]\,,\\
&\Psi[0,1]\Psi[0,1]=\Psi[0,2]+\Psi[1,0]\,,\\
&\Psi[0,1]\Psi[2,0]=\Psi[1,0]+\Psi[1,1]\,,\\
&\Psi[0,1]\Psi[0,2]=\Psi[1,1]\,,\\
&\Psi[0,1]\Psi[1,1]=\Psi[0,1]+\Psi[2,0]\,,\\
&\Psi[2,0]\Psi[2,0]=\Psi[0,2]\,,\\
&\Psi[0,2]\Psi[0,2]=\Psi[2,0]\,,\\
&\Psi[0,2]\Psi[1,1]=\Psi[1,0]\,,\\
&\Psi[1,1]\Psi[1,1]=\Psi[1,1]+\Psi[0,0]\,.
\end{aligned}
\tag{10.4}
$$

The nonvanishing η-coefficients for $k=5$ are

$$\eta[0,0]=6\,,\qquad \eta[1,1]=3\,. \tag{10.5}$$

By using the decomposition (10.1), one finds

$$\langle \tilde{W}(\mathscr{L}(1))\rangle|_{S^3}=6\sqrt{3(\sqrt{5}+1)/2}\,. \tag{10.6}$$

Let us now consider the knot $C\subset T^2\times S^1$ shown in Fig. 10.4; when C has preferred framing and colour ψ_i, one gets

$$\langle \tilde{W}(C;\psi_i)\rangle|_{T^2\times S^1}=\begin{cases}1 & \text{for } \psi_i=\Psi[0,0]\,,\\ 1/2 & \text{for } \psi_i=\Psi[1,1]\,.\end{cases} \tag{10.7}$$

In a generic manifold of the type $\Sigma_g\times S^1$, we can interpret S^1 as a compactified "time" interval, Σ_g being the space-like surface. According to this interpretation, the knot $C\subset T^2\times S^1$ shown in Fig. 10.4 describes the static situation in which a single coloured puncture is present in T^2.

Consider now the static case in which two coloured punctures are present on T^2. Let C_1 and C_2 be the knots in $T^2\times S^1$ which describes these two punctures. From eq. (10.5) it follows that

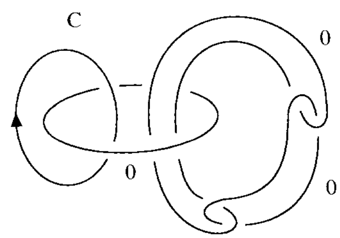


Fig. 10.4. Knot C in $T^2\times S^1$.

$$\langle \tilde{W}(C_1;\psi_i)W(C_2;\psi_j)\rangle|_{T^2\times S^1}=\delta_{ij*}+\tfrac{1}{2}N_{ij[1,1]}\,. \tag{10.8}$$

It is clear that eq. (10.8) can easily be generalized to the case in which several punctures are present in T^2. Indeed, by means of the satellite formulae, we can replace the link associated to these punctures with a single knot. The same method can also be used to analyze the situation in which the genus (number of handles) is greater than one. In fact, according to eq. (10.1), adding one handle is equivalent to the introduction of a puncture with colour $\psi_{(h)}=\sum_i \eta(i)\psi_i$. For example, let us consider $\Sigma_2\times S^1$ with one puncture having colour ψ_i. By using eqs. (10.4) and (10.5) one obtains

$$\langle \tilde{W}(C;\psi_i)\rangle|_{\Sigma_2\times S^1}=\delta_{i[0,0]}+\delta_{i[1,1]}\,. \tag{10.9}$$

11. Three-manifold invariant

As we have shown in the previous sections, the expectation values $\langle W(L)\rangle|_{\mathscr{M}}$ represent topological invariants of the link L in the three-manifold $\mathscr{M}$. We would now like to construct a topological invariant of the manifold $\mathscr{M}$ itself. In the surgery presentation of the manifolds, each three-manifold $\mathscr{M}$ is characterized by a class of equivalent surgery links in S^3. Thus, it is natural to look for a manifold invariant [3] which is defined by the expectation values of the Wilson line operators associated with the surgery links $\{\mathscr{L}\}$. In the derivation of the surgery rules for the field theory, we have seen that $\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}$ carries information on the manifold $\mathscr{M}$. Unfortunately, $\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}$ is not invariant under Kirby moves and, consequently, it cannot represent a three-manifold invariant. In Section 6 we have shown that, under a Kirby move, $\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}$ gets multiplied by the phase factor $e^{\pm i\varphi}$, where $e^{i\varphi}$ is given in eq. (6.5). Therefore, in order to define a topological invariant, we simply need to introduce [7–12] a multiplicative term which cancels out this phase factor.

Let $\mathscr{M}$ be the (closed, connected and orientable) three-manifold corresponding to the "honest" surgery link $\mathscr{L}$ in S^3. Let us introduce an orientation for $\mathscr{L}$ and let us denote by $\sigma(\mathscr{L})$ the signature of the linking matrix associated with $\mathscr{L}$.

Theorem 5. *The quantity*

$$\mathscr{I}(\mathscr{M})=\exp[i\varphi\sigma(\mathscr{L})]\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}\,. \tag{11.1}$$

where $\tilde{W}(\mathscr{L})$ has been defined in eq. (6.6), is invariant under Kirby moves and then represents a three-manifold invariant.

Proof. Under a Kirby move, $\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}$ transforms as

$$\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}\to e^{\pm i\varphi}\langle \tilde{W}(\mathscr{L})\rangle|_{S^3}\,, \tag{11.2}$$

whereas the signature $\sigma(\mathscr{L})$ transforms as [10,12]

$$\sigma(\mathscr{L}) \to \sigma(\mathscr{L}) \mp 1 . \tag{11.3}$$

Therefore $\mathscr{I}(\mathscr{M})$ is invariant under Kirby moves. □

According to the surgery rules (6.12), $\mathscr{I}(\mathscr{M})$ can be interpreted [3] as the value of the (improved) partition function of the Chern–Simons field theory in $\mathscr{M}$.

In the sequel of this section we give the value of $\mathscr{I}(\mathscr{M})$ for various three-manifolds. First of all we note that, according to the definition (11.1), one has

$$\mathscr{I}(S^3) = 1 . \tag{11.4}$$

The manifold $S^2 \times S^1$ admits a surgery presentation described by the unknot with surgery coefficient $r=0$. In this case, $\sigma(\mathscr{L})=0$ and then one gets

$$\mathscr{I}(S^2 \times S^1) = \begin{cases} \sqrt{3} & \text{for } k=1,2 , \\ (k\sqrt{3}/16) \cos^{-1}(\pi/k) \sin^{-3}(\pi/k) & \text{for } k \geqslant 3 . \end{cases} \tag{11.5}$$

Let us consider the lens spaces $L(p,1)$ with $p \geqslant 2$; a surgery link for these manifolds is the unknot with surgery coefficient $r=p$, the signature of the corresponding linking matrix is equal to $+1$. By using the results of Section 9, one has
for $k=1$:

$$\mathscr{I}(L(p,1)) = (\mathrm{i}/\sqrt{3})(1+2\mathrm{e}^{-\mathrm{i}\pi p/3}) ; \tag{11.6}$$

for $k=2$:

$$\mathscr{I}(L(p,1)) = (-\mathrm{i}/\sqrt{3})(1+2\mathrm{e}^{\mathrm{i}\pi p/3}) ; \tag{11.7}$$

for $k=3$:

$$\mathscr{I}(L(p,1)) = 1 ; \tag{11.8}$$

for $k=4$:

$$\mathscr{I}(L(p,1)) = (\mathrm{i}/\sqrt{3})(1+2\mathrm{e}^{-\mathrm{i}\pi p/3}) ; \tag{11.9}$$

for $k=5$:

$$\mathscr{I}(L(p,1)) = \mathrm{e}^{-\mathrm{i}6\pi/5} \sqrt{\frac{2}{3(\sqrt{5}+1)}} \, [1+2\mathrm{e}^{\mathrm{i}2\pi p/3}] \times [1+\tfrac{1}{2}(3+\sqrt{5})\, \mathrm{e}^{-\mathrm{i}6\pi p/5}] . \tag{11.10}$$

The Poincaré manifold admits a surgery presentation described by the right-handed trefoil with surgery coefficient $r=1$. The values of the associated invariant for $k=1, 2, 3, 4, 5$ are
for $k=1, 2, 3, 4$:

$$\mathscr{I}(\mathscr{P})=1\ ; \tag{11.11}$$

for $k=5$:

$$\mathscr{I}(\mathscr{P})=\sqrt{\frac{2}{3(\sqrt{5}+1)}}\,(2-\mathrm{i}\sqrt{3})\,\frac{3+\sqrt{5}}{2}\,(1-\mathrm{e}^{-\mathrm{i}\pi/5})\ . \tag{11.12}$$

The manifold $T^2\times S^1$ corresponds to the surgery link shown in Fig. 10.3. From eq. (10.2), it follows that $\eta(1)=\sum_i 1=\dim \mathscr{T}_{(k)}$. Consequently, one has

$$\mathscr{I}(T^2\times S^1)=\mathscr{I}(S^2\times S^1)\dim \mathscr{T}_{(k)}\ . \tag{11.13}$$

Let us now consider the manifold $\Sigma_g\times S^1$; by using the handle decomposition (10.1), we find
for $k=1, 2, 4$:

$$\mathscr{I}(\Sigma_g\times S^1)=\sqrt{3}\;3^g\ ; \tag{11.14}$$

for $k=3$:

$$\mathscr{I}(\Sigma_g\times S^1)=1\ ; \tag{11.15}$$

for $k=5$:

$$\mathscr{I}(\Sigma_g\times S^1)=\begin{cases}3^g\cdot 5^{g/2}F(g-1)\,\sqrt{3(\sqrt{5}+1)/2} & \text{for } g \text{ even},\\ 3^g\cdot 5^{(g-1)/2}[F(g)+F(g-2)]\sqrt{3(\sqrt{5}+1)/2} & \text{for } g \text{ odd}.\end{cases} \tag{11.16}$$

In eq. (11.16), $F(g)$ denotes the gth Fibonacci number; i.e., $F(g)$ is defined by $F(g)=F(g-1)+F(g-2)$, with $F(1)=F(2)=1$. Details on the derivation of eq. (11.16) can be found in Appendix C.

12. Connections with conformal field theory

The analytic correlation functions of primary fields in the conformal WZNW model possess monodromy properties described by the famous Knizhnik–Zamolodchikov equation [15]. The associated braid group representation has been studied in great details; as shown by Kohno [16], these monodromy representations are equivalent to the R-matrix representations defined in terms of the so-called quantum deformations of the simple Lie algebras. On the other hand, the skein relations satisfied by the expectation values of the CS theory can be associated with braid group representations which, again, are equivalent to the R-matrix representations [17]. Thus, the braiding structures which are found in conformal field theory, in the quantum group approach and in the CS theory coincide. In fact, as shown by Drinfeld, the braid representations determined by the quasitensor category associated with the quasi-triangular quasi-Hopf algebras are universal [18].

Consequently, several algebraic relations which are found in the different models have a universal structure. Of course, these relations may admit different interpretations depending on the particular models which are considered. In particular, as discussed by Witten in [6], the fusion rules of the $SU(3)_l$ WZNW model should have a corresponding counterpart in the SU(3) CS theory. Indeed, the algebra of the fusion rules of the SU(3) WZNW model of level l is isomorphic with the reduced tensor algebra $\mathscr{T}_{(k)}$ of the SU(3) CS theory with $k=l+3$.

The proof is very simple. On the one hand, the Knizhnik–Zamolodchikov equation in the conformal theory determines a flat connection of the type considered by Kohno and the Sugawara for of the energy–momentum tensor implies that the resulting deformation parameter is given by $q=\exp(-i2\pi/(l+3))$. With this value of the deformation parameter, the fusion rules close on a finite set of conformal blocks which are labelled by certain irreducible representations of SU(3).

On the other hand, in the CS theory the deformation parameter is $q=\exp(-i2\pi/k)$. The components of the links are labelled by the irreducible representations of SU(3) and the three-dimensional counterpart of the fusion property of the primary fields is represented by the structure of the satellite relations. By comparing the deformation parameters of the conformal theory and of the CS theory, one finds that they coincide when $k=l+3$. With this fixed integer value of k, we have seen that the algebraic structure associated with the satellite relations of the CS theory close on a finite set of physically inequivalent representations. As we have shown in this paper and in [1], these representations identify the equivalence classes belonging to the reduced tensor algebra $\mathscr{T}_{(k)}$. The structure constants of $\mathscr{T}_{(k)}$ characterize the satellite relations and then they correspond to the fusion rules of the conformal theory. Therefore, because of Drinfeld's universality theorem [18], the fusion algebra of the conformal theory with level l must coincide with the reduced tensor algebra $\mathscr{T}_{(k=l+3)}$.

In the case in which the gauge group is SU(2), the equivalence between the structure constants of the reduced tensor algebra and the fusion rules of the $SU(2)_l$ WZNW model has been proved explicitly [17]. When $G=SU(3)$, it is immediate to verify that, for example, the fusion algebra of level $l=1$ is isomorphic with $\mathscr{T}_{(4)}$. The general properties (5.8)–(5.11), which we have obtained by means of the properties of the CS theory, are exactly the properties [19] that the fusion coefficients must satisfy.

Furthermore, one can show [20] that the S-matrix of the SU(3) WZNW conformal model is strictly connected with the Hopf matrix $H[(m,n);(a,b)]$ given in eq. (8.4) of Ref. [1]. The celebrated property that the S-matrix diagonalize the fusion rules admits a simple interpretation [20] in the CS theory. As well known [21], the dimension of the bundle of the generalized characters on a Riemann surface of genus g can be expressed in terms of the S-matrix. The same formula can also be derived [20] in the CS theory.

13. Conclusions

In this paper we have given the solution of the Chern–Simons theory in any connected, closed and orientable three-manifold $\mathscr{M}$ when the gauge group is SU(3). By using the structure of the reduced tensor algebra $\mathscr{T}_{(k)}$ associated with SU(3), we have derived the field theory surgery rules which permit us to relate the values of the observables in $\mathscr{M}$ with the expectation values in S^3.

We have analyzed also the general features of the surgery rules for a generic simple compact Lie group G. Quite remarkably, the form of the surgery colour state Ψ_0 turns out to be universal; in particular, Ψ_0 is determined by the values of the unknot.

The surgery rules have been used to compute the expectation values of Wilson line operators defined in several nontrivial three-manifolds. We have shown how the improved partition function $\mathscr{I}(\mathscr{M})$ of the theory can be expressed in terms of the expectation values of the Wilson line operators associated with the surgery links. Invariance of $\mathscr{I}(\mathscr{M})$ under Kirby moves has been proved. Various three-manifolds have been considered and the corresponding values of $\mathscr{I}(\mathscr{M})$ have been computed.

Appendix A

The main purpose of this appendix is to show that the structure constants of the reduced tensor algebra verify the relation $N_{ijm}=N_{i*mj}$. We shall also include a discussion on the normalization factors appearing in the surgery colour states. Let us recall that the elements of the standard basis of $\mathscr{T}_{(k)}$ are denoted by $\{\psi_i\}$. The unit element of the algebra is indicated by ψ_1.

Property A.1. *The structure constants of $\mathscr{T}_{(k)}$ satisfy*

$$N_{ij1}=\delta_{ij*}=\delta_{i*j}\ . \tag{A.1}$$

Proof. When $k=1$ and $k=2$, the validity of eq. (A.1) can easily be verified by direct inspection [1]. Let us concentrate then on the case $k\geqslant 3$. In Ref. [1] we have proved that the Hopf matrix H satisfies

$$(H^2)_{ij}=b(k)\delta_{ij*}\ , \tag{A.2}$$

where

$$b(k)=\frac{3k^2}{256\sin^6(\pi/k)\cos^2(\pi/k)}. \quad \text{(A.3)}$$

Since H_{ij} represents the value of the Hopf link, by using the connected sum formula (4.16) of Ref. [1], one finds

$$(H^2)_{ij}=\sum_m E_0[m]\langle W(C_1, C_2, C_3; \psi_i, \psi_j, \psi_m)\rangle|_{S^3}, \quad \text{(A.4)}$$

where $\{C_1, C_2, C_3\}$ are the three (framed) components of the link shown in Fig. A.1. By using the satellite relations (7.1) and eq. (A.2), one obtains

$$(H^2)_{ij}=b(k)N_{ij1}=b(k)\delta_{ij*}, \quad \text{(A.5)}$$

which shows that eq. (A.1) is satisfied. □

Let us introduce the complex-valued linear functional $\mathscr{F}$ which is defined on the elements $\mathscr{T}_{(k)}$. The action of $\mathscr{F}$ on the elements of the standard basis of $\mathscr{T}_{(k)}$ is defined as

$$\mathscr{F}(\psi_i)=\delta_{i1}. \quad \text{(A.6)}$$

From Property A.1, it follows that

$$\mathscr{F}(\psi_i\psi_j)=N_{ij1}=\delta_{ij*}. \quad \text{(A.7)}$$

Consequently, by means of $\mathscr{F}$, the structure constants N_{ijm} can be written as

$$\mathscr{F}(\psi_i\psi_j\psi_{m*})=\sum_n N_{ijn}\mathscr{F}(\psi_n\psi_{m*})=N_{ijm}. \quad \text{(A.8)}$$

Since $\mathscr{T}_{(k)}$ is a commutative and associative algebra, one has

$$N_{ijm}=\mathscr{F}(\psi_i\psi_{m*}\psi_j)=N_{im*j*}. \quad \text{(A.9)}$$

At this point, one can use the relation $N_{ijm}=N_{i*j*m*}$ (see eq. (5.9)); therefore, one finally gets

$$N_{ijm}=N_{i*mj}, \quad \text{(A.10)}$$

which coincides with eq. (5.10).

In order to study the normalization properties of the surgery colour states, let us introduce the following quantities

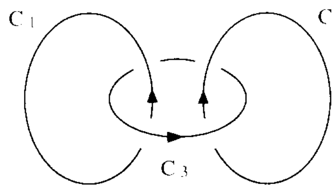


Fig. A.1. Connected sum of two Hopf links.

$$Z_0 = \sum_i E_0^2[i] \,, \tag{A.11}$$

$$Z_{(\pm 1)} = \sum_i q^{\pm Q(i)} E_0^2[i] \,. \tag{A.12}$$

By definition, one has $Z_{(-1)} = Z^*_{(+1)}$, moreover

$$(H^2)_{ij} = Z_0 \delta_{ij^*} \,. \tag{A.13}$$

Property A.2. *The complex numbers* Z_0, $Z_{(+1)}$ *and* $Z_{(-1)}$ *satisfy*

$$Z_0 = Z_{(+1)} Z_{(-1)} \,. \tag{A.14}$$

Proof. On the one hand one obtains from eq. (7.6)

$$\sum_{ij} q^{Q(i)} E_0[i] E_0[j] H_{ij} = Z_{(+1)} Z_{(-1)} \,. \tag{A.15}$$

On the other hand, by using the formula [1] for the connected sum of links, one finds

$$\begin{aligned} \sum_{ij} q^{Q(i)} E_0[i] E_0[j] H_{ij} &= \sum_i q^{Q(i)} E_0[i] (H^2)_{i1} \\ &= Z_0 \sum_i q^{Q(i)} E_0[i] \delta_{i1} \\ &= Z_0 \,. \end{aligned} \tag{A.16}$$

Eq. (A.14) follows from eqs. (A.15) and (A.16). □

Appendix B

The value of Z_0, defined in eq. (A.11), has been computed in Ref. [1]; for $k \geqslant 3$, one has

$$Z_0 = \frac{3k^2}{256 \sin^6(\pi/k) \cos^2(\pi/k)} \,. \tag{B.1}$$

In this appendix we shall compute Z_{+1}, shown in eq. (A.12), when $k \geqslant 3$. Let us recall that the elements of the standard basis of $\mathscr{T}_{(k)}$ are $\{\Psi[m, n]\}$ where the couples (m, n) label the points of the fundamental domain Δ_k,

$$\Delta_k \equiv \{(m, n)\} \quad \text{with} \begin{cases} 0 \leqslant m < k-2 \,, \\ 0 \leqslant n < -m+k-2 \,. \end{cases} \tag{B.2}$$

Since the value of the unknot vanishes on the points which belong to the boundary of Δ_k, eq. (A.12) can be written as

$$Z_{(+1)} = \sum_{m=0}^{k-2} \sum_{n=0}^{k-2-m} q^{Q(m,n)} E_0^2[m,n] . \tag{B.3}$$

By using the correspondence rules given in Ref. [1], it is easy to verify that the sum appearing in eq. (B.3) can be extended to the region $0 \leqslant m \leqslant (4k-1)$ and $0 \leqslant n \leqslant (3k-1)$ provided that we divide by a factor $24 = 4 \times 3 \times 2$. Thus,

$$Z_{(+1)} = \frac{1}{24} \sum_{m=0}^{4k-1} \sum_{n=0}^{3k-1} q^{Q(m,n)} E_0^2[m,n] . \tag{B.4}$$

By inserting in expression (B.4) the values $Q(m, n)$ of the quadratic Casimir operator and the values $E_0[m, n]$ of the unknot, one finds

$$\begin{aligned}
Z_{(+1)} = \frac{1}{24} \sum_{m=0}^{4k-1} \sum_{n=0}^{3k-1} & (1-\mathrm{e}^{-\mathrm{i}(2\pi/k)})^{-6} (1+\mathrm{e}^{-\mathrm{i}(2\pi/k)})^{-2} \\
& \times \{ \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m-3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2-3m)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)(n^2+mn)} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2-3m+3)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2-3m+6)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m-3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3)} \\
& + 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)(n^2+mn)} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+6)} \\
& + 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+9)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+6)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+12)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m-3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3m+6)} \\
& + 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)(n^2+mn)} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3m+9)} \\
& - 6\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3m+12)} \\
& + 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+6)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3m+15)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+9)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+3m+18)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)(n^2+mn)} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+6m+12)} \\
& + 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+3)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+6m+15)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+6)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+6m+18)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+9)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+6m+21)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+3)]} \, \mathrm{e}^{--\mathrm{i}(2\pi/3k)(m^2+9m+18)} \\
& - 2\mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+6)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+9m+21)} \\
& + \mathrm{e}^{-\mathrm{i}(2\pi/3k)[n^2+n(m+9)]} \, \mathrm{e}^{-\mathrm{i}(2\pi/3k)(m^2+9m+24)} \} .
\end{aligned} \tag{B.5}$$

The sums appearing in expression (B.5) have the form of generalized double Gauss sums. In order to evaluate this expression, we need to recall a few properties of the Gauss sums. Let $F(y, \alpha)$ be the function (generalized Gauss sum)

$$F(y, \alpha)=\sum_{n=0}^{y-1} \mathrm{e}^{-(\mathrm{i}2\pi/y)(n^2+\alpha n)}, \tag{B.6}$$

where y (with $y>1$) and α are integers. By using the reciprocity formula [14] for the generalized Gauss sums

$$\sum_{n=0}^{|c|-1} \mathrm{e}^{(\mathrm{i}\pi/c)(an^2+bn)}=\sqrt{|c/a|}\, \mathrm{e}^{(\mathrm{i}\pi/4ac)(|ac|-b^2)} \sum_{n=0}^{|a|-1} \mathrm{e}^{(-\mathrm{i}\pi/a)(cn^2+bn)}, \tag{B.7}$$

one gets

$$F(y, \alpha)=\sqrt{\frac{y}{2}}\, \mathrm{e}^{(\mathrm{i}\pi/4)(2\alpha^2/y-1)}[1+\mathrm{e}^{(\mathrm{i}\pi/2)(y+2\alpha)}]. \tag{B.8}$$

Eq. (B.8) can be used to compute $Z_{(+1)}$. Indeed, each term entering expression (B.5) consists of a double generalized Gauss sum. By using eq. (B.8) twice, each term can be evaluated explicitly. The final result is

$$Z_{(+1)}=\mathrm{e}^{\mathrm{i}6\pi/k} \frac{k\sqrt{3}}{16 \cos(\pi/k) \sin^3(\pi/k)}. \tag{B.9}$$

A nontrivial check of eq. (B.9) is the following. The product $Z_{(+1)}Z_{(-1)}=|Z_{(+1)}|^2$ coincides with Z_0 given in eq. (B.1). This means that the result (B.9) is in agreement with Property A.2.

Appendix C

In this appendix, we give the explicit derivation of eq. (11.15). The manifold $\Sigma_g \times S^1$ can be obtained from $S^2 \times S^1$ by "adding g handles". As we have shown in Section 10, each handle admits the decomposition (10.1). When $k=5$, the non-vanishing values of the η-coefficients are shown in eq. (10.5). Therefore, in order to compute $\mathscr{I}(\Sigma_g \times S^1)$, we need to consider the manifold $S^2 \times S^1$ with g punctures on S^2 where each puncture has colour $\Psi_h=6\Psi[0,0]+3\Psi[1,1]$. We shall decompose the resulting colour state

$$\Psi_{gh}=(\Psi_h)^g=3^g(2\Psi[0,0]+\Psi[1,1])^g, \tag{C.1}$$

and, because of eq. (8.6), we have to find the coefficient $\eta_{gh}(0)$ of $\Psi[0,0]$ in this decomposition. From (10.4) one has

$$\Psi[0,0]\Psi[0,0]=\Psi[0,0], \qquad \Psi[0,0]\Psi[1,1]=\Psi[1,1],$$

$$\Psi[1,1]\Psi[1,1]=\Psi[0,0]+\Psi[1,1]\,. \tag{C.2}$$

Therefore, if Ψ_b denotes the state

$$\Psi_b=\Psi[0,0]+x\Psi[1,1]\,, \tag{C.3}$$

where x is a root of the equation $x^2=x+1$ one gets (for $n\geqslant 2$)

$$(\Psi_b)^n=(x+2)^{n-1}\Psi_b\,. \tag{C.4}$$

The state $2\Psi[0,0]+\Psi[1,1]$ can be written as

$$2\Psi[0,0]+\Psi[1,1]=x^{-1}((2x+1)\Psi[0,0]+\Psi_b)\,. \tag{C.5}$$

Consequently, from eq. (C.1) it follows that

$$\eta_{gh}(0)=\frac{3^g(2x-1)^g}{x^g(x+2)}\left[x+1+\left(1+\frac{x+2}{2x-1}\right)^{g}\right]. \tag{C.6}$$

The two possible values of x are $(1\pm\sqrt{5})/2$. Therefore, eq. (C.6) becomes

$$\eta_{gh}(0)=3^g\cdot 5^{(g-1)/2}\left[\left(\frac{\sqrt{5}-1}{2}\right)^{g-1}+\left(\frac{\sqrt{5}+1}{2}\right)^{g-1}\right]. \tag{C.7}$$

Since the equation $x^2=x+1$ defines the recursive relation

$$x^g=F(g)x+F(g-1) \tag{C.8}$$

for the Fibonacci numbers $\{F(g)\}$, the value of the invariant

$$\mathcal{I}(\Sigma_g\times S^1)=\eta_{gh}(0)\sqrt{3(\sqrt{5}+1)/2} \tag{C.9}$$

can be written in the final form

$$\mathcal{I}(\Sigma_g\times S^1)=\begin{cases}3^g\cdot 5^{g/2}F(g-1)\sqrt{3(\sqrt{5}+1)/2} & \text{for } g \text{ even}\,,\\ 3^g\cdot 5^{(g-1)/2}[F(g)+F(g-2)]\sqrt{3(\sqrt{5}+1)2} & \text{for } g \text{ odd}\,,\end{cases} \tag{C.10}$$

which is the equation reported in Section 11.

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